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New Laplace transforms of Kummer's confluent hypergeometric functions

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ABSTRACT

In this paper we aim to show how one can obtain so far unknown Laplace transforms of three rather general cases of Kummer's confluent hypergeometric function ${}_1F_1(a; b; x)$ by employing generalizations of Gauss's second summation theorem, Bailey's summation theorem and Kummer's summation theorem obtained earlier by Lavoie, Grondin and Rathie. The results established may be useful in theoretical physics, engineering and mathematics.

1. Introduction

In a series of papers, Lavoie et al. [1–3] have obtained the generalizations of various classical summation theorems such as the Kummer, Gauss second and Bailey ones for the ${}_2F_1$ series as well as the Watson, Dixon and Whipple ones for the ${}_3F_2$ series.

In a present investigation, we shall need the following generalizations of summation theorems [3, p. 297]: the generalization of Gauss's second summation theorem

$$ {}_2F_1\left(\left.\begin{matrix} a, b \\ \frac{1}{2}(a+b+i+1) \end{matrix}\right| \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \times \left\{\frac{A_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left\lfloor\frac{1+i}{2}\right\rfloor\right)} + \frac{B_i(a,b)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i-\left\lfloor\frac{i}{2}\right\rfloor\right)}\right\}, \quad (1) $$

the generalization of Bailey's summation theorem

$$ {}_2F_1\left(\left.\begin{matrix} a, 1-a+i \\ b \end{matrix}\right| \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma(1-a)}{2^{b-i-1}\Gamma\left(1-a+\frac{1}{2}i+\frac{1}{2}|i|\right)} $$

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$$\times\left\{\frac{C_i(a,b)}{\Gamma\left(\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2}\right)\left(\frac{1}{2}b+\frac{1}{2}a-\left\lfloor\frac{1+i}{2}\right\rfloor\right)}+\frac{D_i(a,b)}{\Gamma\left(\frac{1}{2}b-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}a-\frac{1}{2}-\left\lfloor\frac{i}{2}\right\rfloor\right)}\right\} \quad (2)$$

and the generalization of Kummer's summation theorem

$$ {}_2F_1\left(\begin{matrix}a,b\\1+a-b+i\end{matrix}\middle|-1\right)=\frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)}\left\{\frac{1}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)}\right.$$

$$\left.\times\frac{E_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\left\lfloor\frac{1+i}{2}\right\rfloor\right)}+\frac{F_i(a,b)}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left\lfloor\frac{i}{2}\right\rfloor\right)}\right\}. \quad (3)$$

Here (and hereafter) it is assumed that $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$. Also, throughout the text, as usual, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. Observe that all the coefficients which appear in Eqs. (1), (2) and (3) are listed in Tables 1–3 and that, for $i = 0$, these equations respectively reduce to Gauss's second, Bailey's and Kummer's summation theorems (see, for instance, [4, Appendix III]).

Table 1
Values of the coefficients $A_i(a, b)$ and $B_i(a, b)$ in Eq. (1).

i	$A_i(a,b)$	$B_i(a,b)$
5	$-[(a+b+6)^2-\frac{1}{2}(b-a+6)(b+a+6)-\frac{1}{4}(b-a+6)^2$ $-11(b+a+6)+\frac{13}{2}(b-a+6)+20]$	$[(a+b+6)^2+\frac{1}{2}(b-a+6)(b+a+6)-\frac{1}{4}(b-a+6)^2$ $-17(b+a+6)-\frac{1}{2}(b-a+6)+62]$
4	$\frac{1}{2}(a+b+1)(a+b-3)-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a-1)$
3	$\frac{1}{2}(b-a+4)-(b+a+4)+3$	$\frac{1}{2}(b-a+4)+(b+a+4)-7$
2	$\frac{1}{2}(b+a+3)-2$	-2
1	-1	1
0	1	0
−1	1	1
−2	$\frac{1}{2}(b+a-1)$	2
−3	$\frac{1}{2}(3a+b-2)$	$\frac{1}{2}(3b+a-2)$
−4	$\frac{1}{2}(a+b-3)(a+b+1)-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a-1)$
−5	$[(b+a-4)^2-\frac{1}{2}(b+a-4)(b-a-4)-\frac{1}{4}(b-a-4)^2$ $+4(b+a-4)-\frac{7}{2}(b-a-4)]$	$[(b+a-4)^2+\frac{1}{2}(b-a-4)(b+a+4)-\frac{1}{4}(b-a-4)^2$ $+8(b+a-4)-\frac{1}{2}(b-a-4)+12]$

Table 2
Values of the coefficients $C_i(a, b)$ and $D_i(a, b)$ in Eq. (2).

i	$C_i(a,b)$	$D_i(a,b)$
5	$-(4b^2-2ab-a^2-22b+13a+20)$	$4b^2+2ab-a^2-34b-a+62$
4	$2(b-2)(b-4)-(a-1)(a-4)$	$-4(b-3)$
3	$a-2b+3$	$a+2b-7$
2	$b-2$	-2
1	-1	1
0	1	0
−1	1	1
−2	b	2
−3	$2b-a$	$a+2b+2$
−4	$2b(b+2)-a(a+3)$	$4(b+1)$
−5	$4b^2-2ab-a^2+8b-7a$	$4b^2+2ab-a^2+16b-a+12$

In [3], the above summation formula (2) (with Table 2) was easily deduced (see [3, pp. 297 and 298, Eq. (6) and Table 3]) as a limiting case of the generalized version of Whipple's summation theorem (see [3, pp. 294–296, Eq. (4) and Tables 1 and 2]) by letting $c \to \infty$. The summation formulae (1) and (3) (and their tables of coefficients, Tables 1 and 3) are not explicitly given in [3] but it is emphasized there that simple consequences of (2), *i.e.* the above formulae (1) and (3), follow at once on noting that the well-known Euler transformations for the Gauss function ${}_2F_1$ [5, p. 60, Theorems 20 and 21] yield (see [3, p. 297])

$$ {}_2F_1\left(\begin{matrix}a,\,1-a+i\\b\end{matrix}\middle|\frac{1}{2}\right)=2^{1+i-b}{}_2F_1\left(\begin{matrix}b-a,\,a+b-i-1\\b\end{matrix}\middle|\frac{1}{2}\right)$$

$$=2^{a}{}_2F_1\left(\begin{matrix}a,\,a+b-i-1\\b\end{matrix}\middle|-1\right).$$

The main objective of this work is to demonstrate how one can obtain so far unknown Laplace transforms of three general cases of Kummer's confluent hypergeometric function ${}_1F_1(a; b; x)$ (see Eqs. (7), (8) and (9)) by employing generalizations of summation theorems given by (1), (2) and (3).

Table 3
Values of the coefficients $E_i(a, b)$ and $F_i(a, b)$ in Eq. (3).

i	$E_i(a, b)$	$F_i(a, b)$
5	$-4(6+a-b)^2+2b(6+a-b)+b^2 22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)-b^2-34(6+a-b)-b+62$
4	$2(a-b+3)(a-b+1)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2+16(a-b-4)-b+12$

2. Three general Laplace transforms of ${}_1F_1(a; b; x)$

For nonnegative integers p and q, the generalized hypergeometric function in a variable (argument) z with p numerator parameters $a_1, \ldots, a_p$ and q denominator parameters $b_1, \ldots, b_q$ is, as usual, defined by means of the hypergeometric series (see [4, Chapters 1 and 2] and [5, Chapters 4, 5 and 7])

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!} \tag{4}$$

whenever this series converges and elsewhere by analytic continuation. Here Γ is the familiar Gamma function and $(.)_m$ stands for the Pochhammer (or shifted factorial) symbol defined for any complex number α and nonnegative integers m by $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$. The series defining ${}_pF_q$ converges for all values of z when $p \le q$. If $p = q+1$, then the series (4) converges when $|z| < 1$, it is absolutely convergent on the unit circle if $\Re(b_1+\cdots+b_q-a_1-\cdots-a_p) > 0$ and it is convergent on the circle $|z| = 1$ except at $z = 1$ if $-1 < \Re(b_1+\cdots+b_q-a_1-\cdots-a_p) \le 0$.

We begin with

$$ g(s) = \mathfrak{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt $$

and so we define the (direct) Laplace transform of a function $f(t)$ of a real variable t as the integral $g(s)$ over a range of the complex parameter s, whenever this integral exists in the Lebesgue sense. For more details, see, for instance, [6] or [7].

In view of the known formula $\int_0^\infty e^{-st} t^{\alpha-1} dt = \Gamma(\alpha) s^{-\alpha}$, valid when $\Re(s) > 0$ and $\Re(\alpha) > 0$, by utilizing (4) with $p \le q$, it is an easy matter to show that the Laplace transform of a generalized hypergeometric function ${}_pF_q$ is

$$ \int_0^\infty e^{-st} t^{\nu-1} {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| \omega t\right) dt = \Gamma(\nu) s^{-\nu} {}_{p+1}F_q\left(\begin{matrix} \nu, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| \frac{\omega}{s}\right), \tag{5}$$

provided that when $p < q$, $\Re(\nu) > 0$, $\Re(s) > 0$ and ω is arbitrary, or when $p = q > 0$, $\Re(\nu) > 0$ and $\Re(s) > \Re(\omega)$. Note that the interchange of order of summation and integration when integrating the left-hand side of (5) with respect to t is justified by the uniform convergence of the series (4). In particular, when $p = q = 1$, for Kummer's (confluent hypergeometric) function (also referred to as the confluent hypergeometric function of the first kind) ${}_1F_1$ (see, for instance, [5, Chapter 7]), we conclude that its Laplace transform

$$ \int_0^\infty e^{-st} t^{b-1} {}_1F_1\left(\begin{matrix} a \\ c \end{matrix}\middle| \omega t\right) dt = \Gamma(b) s^{-b} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| \frac{\omega}{s}\right), \tag{6}$$

is given in terms of the Gauss hypergeometric function ${}_2F_1$, where $\Re(b) > 0$ and $\Re(s) > \max(\Re(\omega), 0)$.

Clearly, since (6) is the most general case, it is desirable to find, as far as possible, less general cases involving various particular values of the parameters a and c. For examples of more than a dozen such known additional Laplace transforms of ${}_1F_1$, see Erdélyi et al. [8, Section 4.22], Oberhettinger and Badi [9, Section 1.17] and Prudnikov et al. [10, Section 3.35]. Below, in (7), (8) and (9), we give three new and very general transforms of ${}_1F_1$; they are not listed in the standard tables of the Laplace transform (*cf.* Erdélyi et al. [8, Section 4.22], Oberhettinger and Badi [9, Section 1.17] and Prudnikov et al. [10, Section 3.35]) and we have failed to find them in the literature.

Indeed, if we set in (6) $\omega = \frac{1}{2}s$ and either $b = i - a + 1$ or $c = \frac{1}{2}(a+b+i+1)$, then the resulting series ${}_2F_1\left(\frac{1}{2}\right)$ on the right-hand side of (6) can be summed by using the summation formula (1) or (2), and, after some algebra, we have the following two general results:

$$ \int_0^\infty e^{-st} t^{b-1} {}_1F_1\left(\begin{matrix} a \\ \frac{1}{2}(a+b+i+1) \end{matrix}\middle| \frac{1}{2} t s\right) dt = s^{-b} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} $$

$$\times \left\{ \frac{A_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left\lfloor\frac{1+i}{2}\right\rfloor\right)} + \frac{B_i(a,b)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i-\left\lfloor\frac{i}{2}\right\rfloor\right)} \right\} \tag{7}$$

provided $\Re(b) > 0$ and $\Re(s) > 0$, and

$$\int_0^\infty e^{-st} t^{-a+i} {}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\frac{1}{2}t\,s\right) dt = s^{a-i-1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(c)\Gamma(1-a)\Gamma(1-a+i)}{2^{c-i-1}\Gamma\left(1-a+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$
$$\times \left\{ \frac{C_i(a,c)}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\left\lfloor\frac{1+i}{2}\right\rfloor\right)} + \frac{D_i(a,c)}{\Gamma\left(\frac{1}{2}c-\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}-\left\lfloor\frac{i}{2}\right\rfloor\right)} \right\} \tag{8}$$

provided $\Re(1-a+i) > 0$ and $\Re(s) > 0$. The coefficients A_i and B_i in (7) are the same as those given in Table 1 while C_i and D_i in (8) can be obtained from Table 2 by simply changing b to c.

Similarly, if we set in (6) $\omega = -s$, then the resulting series ${}_2F_1(-1)$ on the right-hand side of (6) can be summed by using the summation formula (3); thus we have

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1\left(\begin{matrix}a\\1+a-b+i\end{matrix}\middle|-t\,s\right) dt = s^{-b} \frac{2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma(1-b)\Gamma(1+a-b+i)}{\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)}$$
$$\times \left\{ \frac{E_i(a,b)}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\left\lfloor\frac{1+i}{2}\right\rfloor\right)} + \frac{F_i(a,b)}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left\lfloor\frac{i}{2}\right\rfloor\right)} \right\}, \tag{9}$$

provided $\Re(b) > 0$ and $\Re(s) > 0$. The coefficients E_i and F_i are listed in Table 3.

If we take $i = 0$ in (7), (8) and (9), we respectively get

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1\left(\begin{matrix}a\\\frac{1}{2}(a+b+1)\end{matrix}\middle|\frac{1}{2}t\,s\right) dt = s^{-b} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}, \tag{10}$$

$$\int_0^\infty e^{-st} t^{-a} {}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|\frac{1}{2}t\,s\right) dt = s^{a-1} \frac{\Gamma(1-a)\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}, \tag{11}$$

and

$$\int_0^\infty e^{-st} t^{b-1} {}_1F_1\left(\begin{matrix}a\\1+a-b\end{matrix}\middle|-t\,s\right) dt = \frac{s^{-b}2^{-a}\Gamma\left(\frac{1}{2}\right)\Gamma(b)\Gamma(1+a-b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(1+\frac{1}{2}a-b\right)}, \tag{12}$$

where $\Re(b) > 0$, $\Re(s) > 0$ and $\Re(1-a) > 0$. Results (10) and (11) are recorded in [4], while (12) appears to be new.

In conclusion, we remark that all the above results involving ${}_1F_1$ can be rewritten in the representation using the Whittaker function $M_{\kappa,\nu}$ since there exists the following relationship:

$${}_1F_1\left(\begin{matrix}a\\b\end{matrix}\middle|z\right) = e^{z/2} z^{-b/2} M_{(b-2a)/2,(b-1)/2}(z).$$

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