



Values of the derivatives of the cotangent at rational multiples of π [☆]

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ABSTRACT

By elementary arguments, we deduce closed-form expressions for the values of all derivatives of the cotangent function at rational multiples of π . These formulae are considerably simpler than similar ones which were found in a different manner by Kölbig. Also, we show that the values of $\cot^{(n)}(\pi x)$, $n \in \mathbb{N}$, at $x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$ are expressible in terms of the values of the Bernoulli polynomials alone.

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1. Introduction

Recently, Adamchik [2, p. 4, Eq. 26] completely solved a long-standing problem of finding a closed-form expression for the higher derivatives of the cotangent function [2,3,6]. In this sequel to the work of Adamchik, by elementary arguments, we deduce that the values of the derivatives of the cotangent at rational multiples of π can be expressed as finite sums involving the Bernoulli polynomials, the sine or cosine functions and known constants. These formulae are considerably simpler than similar ones which were found in a different manner by Kölbig [6, p. 8, Theorem 4]. Also, as an immediate corollary of this result, we find that the values of $\cot^{(n)}(\pi x)$, $n \in \mathbb{N}$, at $x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$ are expressible in terms of the values of the Bernoulli polynomials alone.

2. Statement of the results

In what follows, we shall use the function given by

$$l_\nu(x) = \sum_{k=1}^{\infty} \frac{(e^{2\pi i x})^k}{k^\nu} = Li_\nu(e^{2\pi i x}) \quad (x \in \mathbb{R}) \quad (1)$$

where

$$Li_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu} \quad (2)$$

$(\nu \in \mathbb{C}, |z| < 1; \Re(\nu) > 0, |z| \leq 1, z \neq 1; \Re(\nu) > 1, |z| \leq 1)$

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is the polylogarithm $Li_\nu(z)$ [5, p. 30, Eq. 1.11 (14)]. Observe that

$$\Re[l_\nu(x)] = \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^\nu} \quad \text{and} \quad \Im[l_\nu(x)] = \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^\nu} \quad (x \in \mathbb{R}). \quad (3)$$

The Bernoulli polynomial of degree n in x , $B_n(x)$, is, as usual, given by the generating function [5, p. 36, Eq. 1.13 (2)]

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi). \quad (4)$$

and $B_n = B_n(0)$ is the n -th Bernoulli number [5, p. 35, Eq. 1.13 (1)].

The Hurwitz (or generalized) zeta function $\zeta(s, a)$ is defined as an analytic continuation of the series [5, p. 24, Eq. 1.10 (1)]

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1; a \neq 0, -1, -2, \dots). \quad (5)$$

Our results are as follows.

Theorem. If n, p and $q \in \mathbb{N}$ and p and q are such that $1 \leq p < q$, then

$$(a) \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=p/q} = (-1)^n \frac{(2\pi q)^{2n-1}}{n} \sum_{s=1}^q B_{2n}(s/q) \cos(s2\pi p/q),$$

$$(b) \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=p/q} = (-1)^{n-1} \frac{2(2\pi q)^{2n}}{2n+1} \sum_{s=1}^q B_{2n+1}(s/q) \sin(s2\pi p/q),$$

where $B_n(x)$ is the Bernoulli polynomial of degree n .

Corollary 1. Assume that $n \in \mathbb{N}$ and let B_n be the n -th Bernoulli number. We have

$$(a) \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/2} = (-1)^n \frac{(4\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/2)] = (-1)^n \frac{(2\pi)^{2n-1}(2^{2n} - 1)}{n} B_{2n};$$

$$(b) \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/3} = \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=2/3} = (-1)^n \frac{(6\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/3)] = (-1)^n \frac{(2\pi)^{2n-1}(3^{2n} - 1)}{2n} B_{2n};$$

$$(c) \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/4} = \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=3/4} = (-1)^n \frac{(8\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/2)] = (-1)^n \frac{(4\pi)^{2n-1}(2^{2n} - 1)}{n} B_{2n};$$

$$(d) \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=1/6} = \frac{d^{2n-1}}{dx^{2n-1}} \cot(\pi x)|_{x=5/6} = (-1)^n \frac{(12\pi)^{2n-1}}{n} [B_{2n} - B_{2n}(1/2) - B_{2n}(1/3) + B_{2n}(1/6)]$$

$$= (-1)^n \frac{(2\pi)^{2n-1}(2^{2n} - 1)(3^{2n} - 1)}{2n} B_{2n}.$$

Corollary 2. Assume that $n \in \mathbb{N}$ and let $B_n(x)$ be the Bernoulli polynomial of degree n . We have

$$(a) \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/2} = 0;$$

$$(b) \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/3} = -\frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=2/3} = (-1)^{n-1} \frac{2(6\pi)^{2n}\sqrt{3}}{2n+1} B_{2n+1}(1/3);$$

$$(c) \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/4} = -\frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=3/4} = (-1)^{n-1} \frac{4(8\pi)^{2n}}{2n+1} B_{2n+1}(1/4);$$

$$(d) \frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=1/6} = -\frac{d^{2n}}{dx^{2n}} \cot(\pi x)|_{x=5/6} = (-1)^{n-1} \frac{2(12\pi)^{2n}\sqrt{3}}{2n+1} [B_{2n+1}(1/6) + B_{2n+1}(1/3)].$$

3. Proof of the results

In our proof of the theorem we shall use some known results:

- (1) (Simpson's series multisection formula) [7, p. 131]. Let $f(x) = \sum_{k=1}^{\infty} a_k x^k$ and let $\omega = e^{2\pi i/q}$ ($i := \sqrt{-1}$; $q \in \mathbb{N}$). Then for any integer p , $1 \leq p \leq q$, we have $q \sum_{k=0}^{\infty} a_{p+qk} x^{p+qk} = \sum_{s=1}^q \omega^{-sp} f(\omega^s x)$.
- (2) (Abel's theorem) [4, p. 148]. Let $f(x) = \sum_{k=1}^{\infty} a_k x^k$, $|x| < 1$. If $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{x \rightarrow 1^-} f(x) = \sum_{k=1}^{\infty} a_k$.

(3) (The Fourier series for $B_n(x)$). Let $B_n(x)$ be the Bernoulli polynomial as defined in (4). We have [5, p. 38, Eq. 1.13 (15)]

$$B_{2n-1}(x) = (-1)^n \frac{2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n-1}} \quad (0 < x < 1, n = 1; 0 \leq x \leq 1, n = 2, 3, \dots), \tag{6}$$

and [5, p. 37, Eq. 1.13 (14)]

$$B_{2n}(x) = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}} \quad (0 \leq x \leq 1, n \in \mathbb{N}). \tag{7}$$

(4) (Reflection formulae for $\zeta(s, a)$) (see [5, p. 44, Eq. 1.16 (4)] in conjunction with [5, p. 45, Eq. 1.17 (9)]) If $n \in \mathbb{N}$ and $0 < x < 1$, then

$$\zeta(2n+1, x) - \zeta(2n+1, 1-x) = \frac{\pi}{(2n)!} \cot(\pi x)^{(2n)}, \tag{8}$$

$$\zeta(2n, x) + \zeta(2n, 1-x) = -\frac{\pi}{(2n-1)!} \cot(\pi x)^{(2n-1)}. \tag{9}$$

In the proof of Corollary 1 we shall need the following relations [1, pp. 805–806, Eqs. 23.1.19–23.1.24]:

$$B_{2n}(0) = B_{2n}(1) = B_{2n}; \tag{10}$$

$$B_{2n}(1/2) = (2^{1-2n} - 1)B_{2n}; \tag{11}$$

$$B_{2n}(1/3) = B_{2n}(2/3) = (1/2)(3^{1-2n} - 1)B_{2n}; \tag{12}$$

$$B_{2n}(1/4) = B_{2n}(3/4) = 2^{-2n}(2^{1-2n} - 1)B_{2n}; \tag{13}$$

$$B_{2n}(1/6) = B_{2n}(5/6) = (1/2)(2^{1-2n} - 1)(3^{1-2n} - 1)B_{2n}. \tag{14}$$

Proof of Theorem. First, we shall show that, for $p, q \in \mathbb{N}$, $1 \leq p \leq q$, and $\Re(\nu) > 1$, the following holds:

$$\zeta(\nu, p/q) = q^{\nu-1} \sum_{s=1}^q \omega^{-sp} Li_{\nu}(\omega^s), \quad \omega = e^{2\pi i/q}. \tag{15}$$

Clearly, by Abel's theorem and the definition of $\zeta(s, a)$ in (5), the relation in (15) follows at once from

$$\sum_{k=0}^{\infty} \frac{q^{k\nu}}{(p+kq)^{\nu}} = \sum_{s=1}^q \omega^{-sp} Li_{\nu}(\omega^s)$$

which is obtained by making use of Simpson's multisection formula on $Li_{\nu}(z)$, $\Re(\nu) > 1$.

Second, let $l_{\nu}(x)$ be given by (1) and let $p, q \in \mathbb{N}$, $1 \leq p \leq q-1$. Then

$$\zeta(\nu, p/q) - \zeta(\nu, 1-p/q) = 2q^{\nu-1} \sum_{s=1}^q \Im[l_{\nu}(s/q)] \sin(s2\pi p/q), \tag{16}$$

$$\zeta(\nu, p/q) + \zeta(\nu, 1-p/q) = 2q^{\nu-1} \sum_{s=1}^q \Re[l_{\nu}(s/q)] \cos(s2\pi p/q). \tag{17}$$

Indeed, starting from (15) we obtain

$$\zeta(\nu, p/q) = q^{\nu-1} \sum_{s=1}^q (\Re[l_{\nu}(s/q)] + i\Im[l_{\nu}(s/q)]) e^{-s2\pi ip/q} = q^{\nu-1} \sum_{s=1}^q [\Phi_1(s) + i\Phi_2(s)] = q^{\nu-1} \sum_{s=1}^q \Phi_1(s) \quad (1 \leq p \leq q), \tag{18}$$

where

$$\Phi_1(s) = \Re[l_{\nu}(s/q)] \cos(s2\pi p/q) + \Im[l_{\nu}(s/q)] \sin(s2\pi p/q),$$

$$\Phi_2(s) = \Im[l_{\nu}(s/q)] \cos(s2\pi p/q) - \Re[l_{\nu}(s/q)] \sin(s2\pi p/q),$$

i.e. we obtain that the imaginary part vanishes. To prove this, note that for $\Phi_2(s)$ we have that $\Phi_2(q) = 0$ and $\Phi_2(q-s) = -\Phi_2(s)$ ($1 \leq s \leq q-1$) since (see Eq. (3))

$$\Re[l_{\nu}(1-x)] = \Re[l_{\nu}(x)] \quad \text{and} \quad \Im[l_{\nu}(1-x)] = -\Im[l_{\nu}(x)],$$

and therefore $\sum_{s=1}^{q-1} \Phi_2(s) = 0$, so Eq. (18) is valid. Now, by a simple trigonometric consideration, from (18) we have

$$\zeta(\nu, 1-p/q) = q^{\nu-1} \sum_{s=1}^q (\Re[l_{\nu}(s/q)] \cos(s2\pi p/q) - \Im[l_{\nu}(s/q)] \sin(s2\pi p/q)) \quad (1 \leq p \leq q-1). \tag{19}$$

Here, the case $p = q$ should be excluded, considering that for $\zeta(s, a)$ we must have $a \neq 0$ (see the definition of $\zeta(s, a)$ in (5)). Finally, by combining (18) and (19) we arrive at the proposed formulae in (16) and (17).

Third, by applying the Fourier expansions in (6) and (7) in conjunction with (3), from (12) and (13) we have

$$\zeta(2n + 1, p/q) - \zeta(2n + 1, 1 - p/q) = (-1)^{n-1} q^{2n} \frac{(2\pi)^{2n+1}}{(2n + 1)!} \sum_{s=1}^q B_{2n+1}(s/q) \sin(s2\pi p/q) \quad (20)$$

$$\zeta(2n, p/q) + \zeta(2n, 1 - p/q) = (-1)^{n-1} q^{2n-1} \frac{(2\pi)^{2n}}{(2n)!} \sum_{s=1}^q B_{2n}(s/q) \cos(s2\pi p/q). \quad (21)$$

Lastly, the assertions of the theorem follow upon comparing the reflection formulae for $\zeta(s, a)$ in (8) and (9) and our expressions in (20) and (21). \square

Proof of Corollaries. It is straightforward to verify that, in view of [1, pp. 804, Eq. 23.1.8],

$$B_m(1 - x) = (-1)^m B_m(x) \quad (m = 0, 1, 2, \dots), \quad (22)$$

the formulae given by Corollary 2 follow directly from the theorem. In order to deduce the formulae given by Corollary 1 we apply the theorem and make use of the identity (22) and the relations (10) through (14). \square

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