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Geodesic equations in the weak field limit of general $f(R)$ gravity theory

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Abstract. In our work we presented the modified field equations generated by action with unspecified function $f(R)$. Assuming spherical symmetry, we used the corresponding static Schwarzschild-like metric in the weak field limit. Also we considered geodesic equations of motion describing orbits and orbital speeds which can be measured in galactic environment. We solved geodesic equations in the case of a power-law $f(R)$ theories, that is we set $f(R) = f_{0n}R^n$.

1. Introduction

The modified theories of gravity have been proposed like alternative approaches to Einstein theory of gravity [1–5]. In this work we consider power-law fourth-order theories of gravity [6, 7]. $f(R)$ gravity is a straightforward extension of General Relativity (GR) where, instead of the Hilbert-Einstein action, linear in the Ricci scalar R, one considers a power-law $f(R) = f_{0n}R^n$ in the gravity Lagrangian [1, 6–11]. In the weak field limit, a gravitational potential may be written as [6, 7]:

$$\Phi(r) = -\frac{GM}{2r}\left[1 + \left(\frac{r}{r_c}\right)^{\beta}\right], \quad \beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2}, \tag{1}$$

where r_c is the scale-length parameter and it is related to the boundary conditions and the mass of the system and β is a universal parameter related to the power n [7]. For the case $n = 1$, we obtain $\beta(n = 1) = 0$, and the GR is recovered.

In this paper we considered geodesic equations for spherically symmetric static (SSS) metric and power-law fourth-order theories of gravity $f(R) = f_{0n}R^n$. In Section 2 we presented basic properties of SSS metric, in Section 3 we presented field equations in unspecified $f(R)$ gravity, while in Section 4 we find geodesic equations in case of power-law fourth-order theories of $f(R)$ gravity, using procedure as proposed in

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references [12–16]. The calculations of orbits and periods in R^n gravity are presented in Section 5, then, Section 6 provides comparison between our numerical results with astronomical observations, while Section 7 is devoted to concluding remarks.

2. General properties in case of SSS metric

We assume metric for static spherical symmetric space [12]:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2, \tag{2}$$

where: $g_{00} = A$, $g_{11} = -B$, $g_{22} = -r^2$, $g_{33} = -r^2\sin^2\theta$, $g^{\mu\mu} = \dfrac{1}{g_{\mu\mu}}$, $g = g_{00}g_{11}g_{22}g_{33} = -ABr^4\sin^2\theta$.
Cristoffel symbols are [12]: $\Gamma^\alpha_{\varepsilon\nu} = \dfrac{1}{2}g^{\alpha\sigma}\left(g_{\sigma\varepsilon,\nu} + g_{\sigma\nu,\varepsilon} - g_{\varepsilon\nu,\sigma}\right)$ and $\Gamma^\alpha_{\mu\alpha,\nu} = \dfrac{\partial\Gamma^\alpha_{\mu\alpha}}{\partial x^\nu}$ and $g_{\mu\nu,\alpha} = \dfrac{\partial g_{\mu\nu}}{\partial x^\alpha}$.
Crystoffel symbols $\Gamma^\alpha_{\varepsilon\nu}$ different from zero are: $\Gamma^1_{00} = \dfrac{1}{2B}\dfrac{dA}{dr}$, $\Gamma^0_{10} = \Gamma^0_{01} = \dfrac{1}{2A}\dfrac{dA}{dr}$, $\Gamma^1_{11} = \dfrac{1}{2B}\dfrac{dB}{dr}$, $\Gamma^1_{22} = -\dfrac{r}{B}$, $\Gamma^1_{33} = -\dfrac{r}{B}\sin^2\theta$, $\Gamma^2_{12} = \Gamma^3_{13} = \dfrac{1}{r}$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{23} = \Gamma^3_{32} = \mathrm{ctg}\theta$.

Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are expressed:

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\varepsilon\nu}\Gamma^\varepsilon_{\mu\alpha} - \Gamma^\varepsilon_{\mu\nu}\Gamma^\alpha_{\varepsilon\alpha}, \quad R_{00} = -\frac{1}{2B}\frac{d^2A}{dr^2} + \frac{1}{4B^2}\frac{dA}{dr}\frac{dB}{dr} + \frac{1}{4AB}\left(\frac{dA}{dr}\right)^2 - \frac{1}{rB}\frac{dA}{dr},$$

$$R_{11} = \frac{1}{2A}\frac{d^2A}{dr^2} - \frac{1}{4A^2}\left(\frac{dA}{dr}\right)^2 - \frac{1}{4AB}\frac{dA}{dr}\frac{dB}{dr} - \frac{1}{rB}\frac{dB}{dr}, \quad R_{22} = \frac{1}{B} + r\frac{1}{2AB}\frac{dA}{dr} - r\frac{1}{2B^2}\frac{dB}{dr} - 1, \quad R_{33} = R_{22}\sin^2\theta,$$

$$R = g^{\mu\nu}R_{\mu\nu} = -\frac{1}{AB}\frac{d^2A}{dr^2} + \frac{1}{2BA^2}\left(\frac{dA}{dr}\right)^2 + \frac{1}{2AB^2}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r^2}\left(1 - \frac{1}{B}\right) + \frac{2}{rB^2}\frac{dB}{dr} - \frac{2}{rAB}\frac{dA}{dr}. \tag{3}$$

3. Field equations and geodesic equations in unspecified $f(R)$ gravity

As an alternative to Einstein-Hilbert action, we assume action in the form: $S = \int d^4x f(R)\sqrt{-g}$, where $f(R)$ is a function of Ricci scalar R. The field equations of unspecified $f(R)$ gravity are [13]: $R_{\mu\nu} - \dfrac{1}{2}g_{\mu\nu}\dfrac{f}{h} = \left(h_{;\mu\nu} - g_{\mu\nu}h^\lambda_{;\lambda}\right)\dfrac{1}{h}$, $R = \dfrac{2f}{h} - \dfrac{3}{h}h^\lambda_{;\lambda}$, $h = \dfrac{df}{dR}$, $h_{;\mu\nu} = \dfrac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \Gamma^\lambda_{\mu\nu}\dfrac{\partial h}{\partial x^\lambda}$, $h^\lambda_{;\lambda} = \dfrac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu h\right)$, $h_{;11} = \dfrac{\partial^2 h}{\partial r^2} - \Gamma^1_{11}\dfrac{\partial h}{\partial r} = \dfrac{\partial^2 h}{\partial r^2} - \dfrac{1}{2B}\dfrac{\partial B}{\partial r}\dfrac{\partial h}{\partial r}$, $h^\lambda_{;\lambda} = \left(-\dfrac{1}{2AB}\dfrac{dA}{dr} + \dfrac{1}{2B^2}\dfrac{dB}{dr} - \dfrac{2}{rB}\right)\dfrac{dh}{dr} - \dfrac{1}{B}\dfrac{\partial^2 h}{\partial r^2}$, where $h_{;\lambda}$ is covariant derivate.
After some mathematical manipulation given in paper by Sobouti [13] we obtain four field equations [13]:

$$\begin{gathered}
\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr} = -\frac{r}{h}\frac{d^2h}{dr^2} + \frac{r}{2h}\frac{dh}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right), \\
\frac{1}{A}\frac{d^2A}{dr^2} - \frac{1}{2}\left(\frac{1}{A}\frac{dA}{dr} + \frac{2}{r}\right)\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{2}{r^2} + \frac{2B}{r^2} = \frac{2}{h}\frac{d^2h}{dr^2} - \left(\frac{1}{B}\frac{dB}{dr} + \frac{2}{r}\right)\frac{1}{h}\frac{dh}{dr}, \\
\frac{1}{A}\frac{d^2A}{dr^2} - \frac{1}{2A^2}\left(\frac{dA}{dr}\right)^2 - \frac{1}{2AB}\frac{dA}{dr}\frac{dB}{dr} - \frac{2}{r}\frac{1}{B}\frac{dB}{dr} = -B\frac{f}{h} - 2\left(\frac{1}{2A}\frac{dA}{dr} + \frac{2}{r}\right)\frac{1}{h}\frac{dh}{dr}, \\
R = \frac{2f}{h} + \frac{3}{B}\left[\left(\frac{1}{2A}\frac{dA}{dr} - \frac{1}{2B}\frac{dB}{dr} + \frac{2}{r}\right)\frac{1}{h}\frac{dh}{dr} + \frac{1}{h}\frac{d^2h}{dr^2}\right].
\end{gathered} \tag{4}$$

4. Geodesic equations in R^n gravity

We are solving relativistic equations of motion for massive particles in R^n gravity with assumption given in the paper by Capozziello et al. [7]: $AB = 1, h \simeq 1$.

Geodesic equations for the metric (2) are: $\dfrac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{dp}\dfrac{dx^\beta}{dp} = 0$. These equations provide differential equations for the four space-time components: $x^\mu = (t(p), r(p), \theta(p), \varphi(p))$, where p is parameter describing the trajectory. Since, the metric is symmetric about $\theta_0 = \dfrac{\pi}{2}$, the coordinate system may be oriented so that the orbit of the particle lies in that plane, and fix the $\theta = \dfrac{\pi}{2}$ [12]. These equations become:

$$\frac{d^2t}{dp^2} + \frac{1}{A}\frac{dA}{dr}\frac{dr}{dp}\frac{dt}{dp} = 0, \quad \frac{d^2r}{dp^2} + \frac{1}{2B}\frac{dA}{dr}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2B}\frac{dB}{dr}\left(\frac{dr}{dp}\right)^2 - \frac{r}{B}\left(\frac{d\varphi}{dp}\right)^2 = 0, \quad \frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp} = 0. \tag{5}$$

From the first equation we get: $\dfrac{dt}{dp} = \dfrac{1}{A}$. From the third equation we obtain: $J = r^2\dfrac{d\varphi}{dp} = const. = \sqrt{GML} = \sqrt{GMa(1-e^2)}$, and using the second equation we finally have:

$$\left(\frac{dr}{d\varphi}\right)^2 + \frac{r^2}{B}\left(1 + \frac{Er^2}{J^2}\right) = \frac{c^2r^4}{ABJ^2}, \quad \left(\frac{dr}{d\tau}\right)^2 = -\frac{c^2}{B} + \frac{c^4}{ABE} - \frac{c^2J^2}{Er^2B}, \quad \left(\frac{dr}{dt}\right)^2 = -\frac{A^2}{B}E + \frac{Ac^2}{B} - \frac{J^2A^2}{r^2B}, \tag{6}$$

where E and J are constants of the motion and τ is proper time [12].

Also, $ds^2 = c^2d\tau^2 = Edp^2$, where angle $\varphi(r)$ is given by expression: $\varphi(r) = \varphi(r_-) + \displaystyle\int_{r_-}^{r} \frac{\sqrt{B}dr}{r^2\sqrt{-\frac{E}{J^2} + \frac{Bc^2}{J^2} - \frac{1}{r^2}}}$,

and $r_\pm = (1 \pm e)\,a \ \wedge \ L = \left(1 - e^2\right)a$, where a - semimajor axis, L - semilatus rectum, e - eccentricity. The angle of orbital precession per revolution is [12]: $\Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi$.

In case of R^n gravity, taking into account the following equations: $A = 1 + \dfrac{2\Phi}{c^2}$ and $\Phi = -\dfrac{GM}{2r}\left[1 + \left(\dfrac{r}{r_c}\right)^\beta\right]$, we obtain expressions for functions A and B in R^n gravity: $A = 1 - \dfrac{GM}{rc^2}\left[1 + \left(\dfrac{r}{r_c}\right)^\beta\right]$, $B = 1/A$.

We also obtained angular velocity ω in R^n gravity: $\omega = \dfrac{d\varphi}{dt} = \dfrac{JA}{r^2} = \dfrac{J}{r^2} - \dfrac{JGM}{r^3c^2} - \dfrac{JGMr^\beta}{r^3c^2r_c^\beta}$, and orbital velocity:

$$\frac{dr}{dt} = A\sqrt{c^2 - A\left(E + \frac{J^2}{r^2}\right)} = v_{orb}. \tag{7}$$

4.1. *The case of Newtonian limit*

In polar coordinates (r, φ), and with respect to the center of mass, we obtain the following EoM:

$$\frac{d^2r}{dt^2} = -\nabla\Phi(r), \quad \frac{d}{dt}\left[r^2\frac{d\varphi}{dt}\right] = 0 \Rightarrow r^2\frac{d\varphi}{dt} = J = const. \tag{8}$$

The total energy of the system can be written using the reduced mass μ [14]:

$$E_u = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] - \frac{GmM}{r}, \quad \mu = \frac{mM}{m+M}, \quad m \ll M \Rightarrow \mu \approx m, \tag{9}$$

$$\frac{2E_u}{m} = \left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] - \frac{2GM}{r}, \quad \frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\frac{J}{r^2}, \quad \varphi(r) = \int_{r_-}^{r} \frac{dr}{r^2\sqrt{\frac{2E_u}{mJ^2} + \frac{2GM}{J^2 r} - \frac{1}{r^2}}}. \tag{10}$$

It can be shown [17] that the angle of orbital precession per revolution in Newtonian case is: $\Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi = 0$.

4.2. *The case $\beta = 0$ or the case of Schwarzschild metric*

In order to calculate φ and $\Delta\varphi$ to first order in MG/r we need $B(r)$ to the second order, whereas $A(r)$ will be needed only to first order [12]. After mathematical manipulations we obtain following relations:

$$A = 1 - \frac{2GM}{rc^2}, \quad B = 1 + \frac{2GM}{rc^2} + \frac{4G^2M^2}{r^2c^2}, \quad \Delta\varphi\,(0) = 6\pi\frac{G^2M^2}{J^2c^2}, \tag{11}$$

$$\frac{1}{r} = \frac{1 + e\,(0)\cos\left[\varphi\left[1 - \frac{3G^2M^2}{J^2c^2}\right]\right]}{L\,(0)}, \quad \frac{e\,(0)^2}{L\,(0)^2} = \frac{c^2 - E}{J_0^2} + \frac{G^2M^2}{J_0^4}, \tag{12}$$

$$\frac{1}{L\,(0)} = \frac{GM}{J_0^2}, \quad 2\varepsilon\,(0) = c^2 - E, \quad J_0^2 = J^2\left(1 - \frac{4G^2M^2}{c^2J^2}\right), \quad L\,(0) = L - 2r_s, \tag{13}$$

where $r_s = \dfrac{2GM}{c^2}$ (r_s - Schwarzschild radius).

5. The calculations of orbits and periods in R^n gravity

We consider a test particle bound in an orbit around the massive central object. Test particle reaches its minimum and maximum values r_- and r_+ at periapsis and apoapsis, respectively. At both points $dr/d\varphi$ vanishes, so we obtain equations:

$$\frac{dr}{d\varphi}\,(r_\pm) = 0 \Rightarrow \frac{1}{{r_\pm}^2} - \frac{c^2}{J^2A(r_\pm)} = -\frac{E}{J^2}. \tag{14}$$

From these two equations we obtain two constants of motion:

$$\frac{J^2}{c^2} = \frac{\frac{1}{A\,(r_+)} - \frac{1}{A\,(r_-)}}{\frac{1}{{r_+}^2} - \frac{1}{{r_-}^2}} = \frac{{r_+}^2{r_-}^2\,(A\,(r_-) - A\,(r_+))}{A(r_+)A(r_-)\,({r_-}^2 - {r_+}^2)}, \quad \frac{E}{c^2} = \frac{\frac{{r_+}^2}{A(r_+)} - \frac{{r_-}^2}{A(r_-)}}{{r_+}^2 - {r_-}^2} = \frac{A\,(r_-)\,{r_+}^2 - A(r_+){r_-}^2}{A(r_+)A(r_-)\,({r_+}^2 - {r_-}^2)}. \tag{15}$$

After integration of expression (7) and taking into account constants of motion (15) we obtain the period of revolution and the angle of orbital precession per revolution in R^n gravity given by the Eq. (16):

$$T = \frac{2}{J}\int_{r_-}^{r_+} \frac{\sqrt{B^3}dr}{\sqrt{-\frac{E}{J^2} + \frac{Bc^2}{J^2} - \frac{1}{r^2}}} = \frac{2}{J}\int_{r_-}^{r_+} \frac{dr}{A\sqrt{-\frac{AE}{J^2} + \frac{c^2}{J^2} - \frac{A}{r^2}}},$$

$$\varphi(r_+) - \varphi(r_-) = \pm\int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{-\frac{AE}{J^2} + \frac{c^2}{J^2} - \frac{A}{r^2}}} \quad \wedge \quad \Delta\varphi = 2|\varphi(r_+) - \varphi(r_-)| - 2\pi. \tag{16}$$

5.1. The case $\beta = 0$

Here we have: $r_\pm = \dfrac{L(0)}{1 \mp e(0)}$, $\sqrt{B^3} = c_1 + \dfrac{c_2}{r}$, $\dfrac{1}{r} = \dfrac{1 + e(0)\cos k}{L(0)}$. Let us mention here that $e(\beta = 0) = e(0)$ is given by Eqs. (12). Now, we solve integrals I_1 and I_2 [18]:

$$I_1 = \int_0^k \frac{c_2 L(0)}{(1 + e(0)\cos k)} dk = \int_{r_-}^{r} \frac{dr}{r\sqrt{\left(\frac{e(0)}{L(0)}\right)^2 - \left(\frac{1}{r} - \frac{1}{L(0)}\right)^2}} = \frac{2c_2 L(0)}{\sqrt{1 - e(0)^2}} \arctan\left[\frac{1 - e(0)}{\sqrt{1 - e(0)^2}} \tan\frac{k}{2}\right], \quad (17)$$

$$I_2 = \int_0^k \frac{c_1 L(0)^2}{(1 + e(0)\cos k)^2} dk = \int_{r_-}^{r} \frac{dr}{\sqrt{\left(\frac{e(0)}{L(0)}\right)^2 - \left(\frac{1}{r} - \frac{1}{L(0)}\right)^2}} = -\frac{1}{1 - e(0)^2}\left[\frac{c_1 L(0)^2 e(0)\sin k}{1 + e(0)\cos k} - I_1\right], \quad (18)$$

$$Jt = \left(1 + \frac{r_s}{L}\right)(I_1(k) + I_2(k)), \quad J\left(\frac{T}{2} - 0\right) = \left(1 + \frac{r_s}{L}\right)\Big(I_1(k = \pi) + I_2(k = \pi) - I_1(k = 0) - I_2(k = 0)\Big). \quad (19)$$

The period of revolution in case $\beta = 0$ is given by the following expression:

$$T = \frac{2}{\sqrt{GML}}\left(1 + \frac{r_s}{L}\right)\left[L(0)^2 \frac{\pi}{\sqrt{\left(1 - e(0)^2\right)^3}} + \frac{3r_s}{2} L(0) \frac{\pi}{\sqrt{1 - e(0)^2}}\right]. \quad (20)$$

6. Comparison between calculations and some astronomical observations

In this section we compare our calculations with some astronomical observations for S-stars. Tables 1, 2 and 3 present period of revolution (T) and orbital precession ($\Delta\varphi$) for S-stars (S2, S38 and S55), estimated for the following three values of β: $\beta = 0.00001$, $\beta = 0.001$ and $\beta = 0.01$. Value for parameter r_c is taken to be 1, 10^2 and 10^4 AU, respectively. The observed orbital elements and their uncertainties are taken from Table 3 of [19]:

S2: $a = 1044.2 \pm 7.5$ (AU); $e = 0.8839 \pm 0.0019$; $P_{obs} = 16.00 \pm 0.02$ (yr);
S38: $a = 1178.1 \pm 1.7$ (AU); $e = 0.8201 \pm 0.0007$; $P_{obs} = 19.2 \pm 0.02$ (yr);
S55: $a = 896.9 \pm 8.3$ (AU); $e = 0.7209 \pm 0.0077$, $P_{obs} = 12.80 \pm 0.11$ (yr).

Recently, the GRAVITY Collaboration claimed that they detected orbital precession of the S2 star around the Galactic Center [20] and found that it is close to the corresponding GR prediction which for S2 star is $\Delta\varphi = 0^\circ.201$ per orbital period. Also, according to data analysis in the framework of Yukawa gravity model in the paper [21], the orbital precessions of the S38 and S55 stars are close to the corresponding prediction of GR for these stars, which are $0^\circ.119$ and $0^\circ.106$ per orbital period, respectively.

Table 1: Period of revolution (T) in (yr.) and orbital precession ($\Delta\varphi$) in (° per orbital period) for S-stars (S2, S38 and S55), estimated for the following three values of β: $\beta = 0.00001$, $\beta = 0.001$ and $\beta = 0.01$. Value for parameter r_c is taken to be 1 AU. The observed orbital elements and their uncertainties are taken from Table 3 of [19].

Name	Period of revolution (T) (in yr.)			Precession ($\Delta\varphi$) (in °)		
of star	$\beta = 0.00001$	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.00001$	$\beta = 0.001$	$\beta = 0.01$
S2	16.04	16.01	15.77	0.189	0.078	-1.045
S38	19.43	19.40	19.10	0.107	0.004	-1.049
S55	12.91	12.89	12.69	0.096	0.00002	-0.978

From the Tables 1, 2 and 3 we can see that period of revolution and orbital precession for S-stars (S2, S38 and S55) are in good agreement with astronomical observations for very small values of gravitational

Table 2: The same as in Table 1, but value for parameter r_c is taken to be 10^2 AU.

Name of star	Period of revolution (T) (in yr.)			Precession ($\Delta\varphi$) (in °)		
	$\beta = 0.00001$	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.00001$	$\beta = 0.001$	$\beta = 0.01$
S2	16.04	16.03	15.96	0.189	0.079	-1.022
S38	19.43	19.42	19.32	0.108	0.004	-1.026
S55	12.91	12.90	12.84	0.096	0.00002	-0.957

Table 3: The same as in Table 1, but value for parameter r_c is taken to be 10^4 AU.

Name of star	Period of revolution (T) (in yr.)			Precession ($\Delta\varphi$) (in °)		
	$\beta = 0.00001$	$\beta = 0.001$	$\beta = 0.01$	$\beta = 0.00001$	$\beta = 0.001$	$\beta = 0.01$
S2	16.04	16.05	16.14	0.189	0.078	-0.999
S38	19.43	19.44	19.55	0.107	0.004	-1.002
S55	12.91	12.92	12.99	0.096	0.00002	-0.935

parameter $\beta < 0.001$. For larger value of $\beta > 0.001$ precession takes negative sign, i.e. it is opposite to the GR precession. Gravitational parameter r_c has smaller influence on period of revolution and orbital precession for S-stars (S2, S38 and S55) and values of r_c are in the range from 1 to 10^4 AU, which is in agreement with our earlier findings [10, 11].

7. Conclusions

In this work we presented the modified field equations and solved geodesic equations in the case of a power-law $f(R)$ theories, i.e. $f(R) = f_{0n}R^n$. We assume spherical symmetry and we used the corresponding static Schwarzschild-like metric in the weak field limit. Also, using geodesic equations of motion we describe the stellar orbits around Galactic Center, which are measured by observational facilities. We obtain for $\beta = 0$ that the GR is recovered. We show that both parameters β and r_c affect the obtained orbital periods and precessions of S-stars. However, for the studied range of parameters, the influence of β is more noticeable.

Also, our calculations showed a good agreement with the corresponding astronomical observations of several S-stars. We hope that using this method with geodesics, we can evaluate parameters of alternative models for a gravitational potential at the Galactic Center with higher accuracy.

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