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Exponential and trigonometric sums associated with the Lerch zeta and Legendre chi functions

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ABSTRACT

It was shown that numerous (known and new) results involving various special functions, such as the Hurwitz and Lerch zeta functions and Legendre chi function, could be established in a simple, general and unified manner. In this way, among others, we recovered the Wang and Williams–Zhang generalizations of the classical Eisenstein summation formula and obtained their previously unknown companion formulae.

1. Introduction and preliminaries

In a recent paper by Cvijović and Srivastava [1] it was shown that numerous (known or new) results involving various special functions, such as the Hurwitz zeta function, Lerch zeta function and Legendre chi function, could be established in a more general context. The main objective of this sequel is to consider, in a general and unified manner, other seemingly disparate and widely scattered results of this type [2–9], like, for instance, the Wang and Williams–Zhang generalizations of the classical Eisenstein summation formula. In doing so, we have obtained several new results.

The Bernoulli polynomials and numbers, $B_n(x)$ and B_n, are defined by ([5, p. 59]; for generalizations, see [10,11]):

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi)\quad\text{and}\quad B_n:=B_n(0)\quad(n\in\mathbb{N}_0:=\mathbb{N}\cup 0;\ \mathbb{N}:=\{1,2,3,\ldots\}).$$

The Hurwitz and Riemann zeta functions are given by [5, p. 88 *et seq.*]:

$$\zeta(s,a):=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\quad\text{and}\quad\zeta(s)=\zeta(s,1)\quad(a\notin\mathbb{Z}_0^-:=\{0,-1,-2,-3,\ldots\};\Re(s)>1).\tag{1.1}$$

We also use the Lerch (or periodic) zeta function [5, p. 89]:

$$\ell_s(\xi):=\sum_{n=1}^{\infty}\frac{e^{2n\pi i\xi}}{n^s}\quad(i:=\sqrt{-1};\ \xi\in\mathbb{R};\Re(s)>1)\tag{1.2}$$

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and the Legendre chi $\chi_s(z)$ (see, for instance, [12]):

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s} \quad (|z| \le 1; \Re(s) > 1). \tag{1.3}$$

It should be kept in mind that the functions given by (1.1)–(1.3) may be extended by analytic continuation on s. The Hurwitz and Riemann zeta functions, $\zeta(s, a)$ and $\zeta(s)$, are meromorphic in $s \in \mathbb{C}$, with a sole simple pole at $s = 1$. If ξ is not an integer, $\ell_s(\xi)$ is an entire function in $s \in \mathbb{C}$, and for an integer ξ it reduces to $\zeta(s)$. Similarly, the Legendre chi function $\chi_s(z)$ is meromorphic with simple pole at $s = 1$.

2. Statement of main results

Note that, throughout the text, we set an empty sum to be zero and it is assumed that n, p, q and r are positive integers. Our main results are as follows.

Theorem 1. *In terms of the Bernoulli polynomials and the Lerch zeta function, $B_n(x)$ and $\ell_s(\xi)$, we have:*

$$-q^{n-1}\frac{1}{n}B_n\left(\frac{p}{q}\right) = \frac{1}{q}\sum_{r=1}^{q} \ell_{1-n}\left(\frac{r}{q}\right) \mathrm{e}^{-\frac{2\pi \mathrm{i} rp}{q}} \quad (p = 1, \dots, q), \tag{2.1}$$

and

$$\ell_{1-n}\left(\frac{r}{q}\right) = -q^{n-1}\frac{1}{n}\sum_{p=1}^{q} B_n\left(\frac{p}{q}\right) \mathrm{e}^{\frac{2\pi \mathrm{i} pr}{q}} \quad (r = 1, \dots, q). \tag{2.2}$$

Corollary 1A. *We have:*

$$\frac{1}{2} - qB_1\left(\frac{p}{q}\right) = \sum_{r=1}^{q-1} \mathrm{e}^{-\frac{2\pi \mathrm{i} rp}{q}} \left[-\frac{1}{2} + \frac{\mathrm{i}}{2}\cot\left(\frac{\pi r}{q}\right)\right] \quad (p = 1, \dots, q) \tag{2.3}$$

and

$$\frac{1}{2} - \frac{\mathrm{i}}{2}\cot\left(\frac{\pi r}{q}\right) = \sum_{p=1}^{q} \mathrm{e}^{\frac{2\pi \mathrm{i} pr}{q}} B_1\left(\frac{p}{q}\right) \quad (r = 1, \dots, q-1). \tag{2.4}$$

Corollary 1B. *If $n \ge 2$, then, in terms of the Bernoulli polynomials and the derivatives of the cotangent function, we have:*

$$\frac{1}{n}\left[B_n - q^n B_n\left(\frac{p}{q}\right)\right] = \frac{\mathrm{i}}{2(2\pi \mathrm{i})^{n-1}} \sum_{r=1}^{q-1} \mathrm{e}^{-\frac{2\pi \mathrm{i} rp}{q}} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}} \cot(\pi\xi)\bigg|_{\xi=r/q} \quad (p = 1, \dots, q), \tag{2.5}$$

and

$$\frac{\mathrm{i}}{2(2\pi \mathrm{i})^{n-1}} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}} \cot(\pi\xi)\bigg|_{\xi=r/q} = -q^{n-1}\frac{1}{n}\sum_{p=1}^{q} \mathrm{e}^{\frac{2\pi \mathrm{i} pr}{q}} B_n\left(\frac{p}{q}\right) \quad (r = 1, \dots, q-1). \tag{2.6}$$

Remark 1 (*Eisenstein Summation Formula*). Observe that, since $B_1(x) = x - \frac{1}{2}$, the formula (2.3) is equivalent to

$$\sum_{r=1}^{q-1} \sin\left(\frac{2\pi rp}{q}\right)\cot\left(\frac{\pi r}{q}\right) = -2qB_1\left(\frac{p}{q}\right) = q - 2p \quad (p = 1, \dots, q), \tag{2.3*}$$

which is the classical Eisenstein summation formula (see, for instance, [6, p. 360, Eq. (1.8)]), so that the sums in (2.1) as well as in (2.5) can be seen as its generalization.

Remark 2 (*Wang Sums*). The formula (2.1), by means of (3.5) in conjunction with $\ell_s(1) = \zeta(s)$, could be rewritten as follows:

$$\sum_{r=1}^{q-1} \ell_{1-n}\left(\frac{r}{q}\right) \mathrm{e}^{-\frac{2\pi \mathrm{i} rp}{q}} = \frac{1}{n}\left[B_n - q^n B_n\left(\frac{p}{q}\right)\right] \quad (p = 1, \dots, q). \tag{2.1*}$$

In addition, in view of the fact that $B_{2n+1} = 0$, it is clear that (2.5) could be written in the form:

$$\sum_{r=1}^{q-1} \cos\left(\frac{2\pi rp}{q}\right) \frac{\mathrm{d}^{2n-1}}{\mathrm{d}\xi^{2n-1}} \cot(\pi\xi)\bigg|_{\xi=r/q} = (-1)^n \frac{(2\pi)^{2n-1}}{n} \left[q^{2n} B_{2n}\left(\frac{p}{q}\right) - B_{2n}\right] \quad (p = 1, \ldots, q) \tag{2.5*a}$$

and

$$\sum_{r=1}^{q-1} \sin\left(\frac{2\pi rp}{q}\right) \frac{\mathrm{d}^{2n}}{\mathrm{d}\xi^{2n}} \cot(\pi\xi)\bigg|_{\xi=r/q} = (-1)^{n-1} \frac{2(2\pi)^{2n}}{2n+1} q^{2n+1} B_{2n+1}\left(\frac{p}{q}\right) \quad (p = 1, \ldots, q-1). \tag{2.5*b}$$

Observe that our formulae (2.1), (2.3) and (2.5), in the form given by (2.1*), (2.3*), (2.5*a) and (2.5*b), were established by Wang [3, p. 12, Theorems D and C].

Theorem 2. *In terms of the Bernoulli polynomials and the Legendre chi function, $B_n(x)$ and $\chi_s(z)$, we have:*

$$-(2q)^{n-1} \frac{1}{n} B_n\left(\frac{2p-1}{2q}\right) = \frac{1}{q} \sum_{r=1}^{q} \chi_{1-n}\left(\mathrm{e}^{\frac{\pi \mathrm{i} r}{q}}\right) \mathrm{e}^{-\frac{\pi \mathrm{i} r(2p-1)}{q}} \quad (p = 1, \ldots, q), \tag{2.7}$$

and

$$\chi_{1-n}\left(\mathrm{e}^{\frac{\pi \mathrm{i} r}{q}}\right) = -(2q)^{n-1} \frac{1}{n} \sum_{p=1}^{q} B_n\left(\frac{2p-1}{2q}\right) \mathrm{e}^{\frac{\pi \mathrm{i}(2p-1)r}{q}} \quad (r = 1, \ldots, q). \tag{2.8}$$

Corollary 2. *In terms of the Bernoulli polynomials and the derivatives of the cosecant function, we have:*

$$\frac{1}{n}\left[B_n\left(\frac{1}{2}\right) - q^n B_n\left(\frac{2p-1}{2q}\right)\right] = \frac{\mathrm{i}}{2(2\pi \mathrm{i})^{n-1}} \sum_{r=1}^{q-1} \mathrm{e}^{-\frac{\pi \mathrm{i} r(2p-1)}{q}} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}} \csc(\pi\xi)\bigg|_{\xi=r/q} \quad (p = 1, \ldots, q), \tag{2.9}$$

and

$$\frac{\mathrm{i}}{2(2\pi \mathrm{i})^{n-1}} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}} \csc(\pi\xi)\bigg|_{\xi=r/q} = -q^{n-1} \frac{1}{n} \sum_{p=1}^{q} \mathrm{e}^{\frac{\pi \mathrm{i}(2p-1)r}{q}} B_n\left(\frac{2p-1}{2q}\right) \quad (r = 1, \ldots, q-1). \tag{2.10}$$

Remark 3 (*Trigonometric Derivative Formulae*). Observe that the derivative formulae given in (2.6) and (2.10) above were recently derived by Cvijović (see [8, Theorem] and [9, Theorem 1 and Remark 1]). The formula (2.6) could be written in the form below:

$$\frac{\mathrm{d}^{2n-1} \cot(\pi\xi)}{\mathrm{d}\xi^{2n-1}}\bigg|_{\xi=r/q} = \frac{(-1)^n (2q\pi)^{2n-1}}{n} \sum_{p=1}^{q} B_{2n}\left(\frac{p}{q}\right) \cos\left(\frac{2p\pi r}{q}\right) \tag{2.6*a}$$

and

$$\frac{\mathrm{d}^{2n} \cot(\pi\xi)}{\mathrm{d}\xi^{2n}}\bigg|_{\xi=r/q} = \frac{(-1)^{n-1} 2(2q\pi)^{2n}}{2n+1} \sum_{p=1}^{q} B_{2n+1}\left(\frac{p}{q}\right) \sin\left(\frac{2p\pi r}{q}\right). \tag{2.6*b}$$

Similarly, starting from (2.10), we obtain:

$$\frac{\mathrm{d}^{2n-1} \csc(\pi\xi)}{\mathrm{d}\xi^{2n-1}}\bigg|_{\xi=r/q} = \frac{(-1)^n (2q\pi)^{2n-1}}{n} \sum_{p=1}^{q} B_{2n}\left(\frac{2p-1}{2q}\right) \cos\left(\frac{\pi r(2p-1)}{q}\right) \tag{2.10*a}$$

and

$$\frac{\mathrm{d}^{2n} \csc(\pi\xi)}{\mathrm{d}\xi^{2n}}\bigg|_{\xi=r/q} = \frac{(-1)^{n-1} 2(2q\pi)^{2n}}{2n+1} \sum_{p=1}^{q} B_{2n+1}\left(\frac{2p-1}{2q}\right) \sin\left(\frac{\pi r(2p-1)}{q}\right). \tag{2.10*b}$$

Remark 4 (*New Sums*). Clearly, the formulae contained in our Theorem 2 and Corollary 2 may be seen as companions to those in Theorem 1 and Corollaries 1A and 1B. Thus, the following finite sum

$$\sum_{r=1}^{q-1} \chi_{1-n}\left(\mathrm{e}^{\frac{\pi \mathrm{i} r}{q}}\right) \mathrm{e}^{-\frac{\pi \mathrm{i} r(2p-1)}{q}} = \frac{2^{n-1}}{n}\left[B_n\left(\frac{1}{2}\right) - q^n B_n\left(\frac{2p-1}{2q}\right)\right] (p = 1, \ldots, q), \tag{2.7*}$$

which is obtained from (2.7) by making use of (3.5), (1.1) and $\chi_s(1) = (1 - 2^{-s})\zeta(s)$, as well as

$$\sum_{r=1}^{q-1} \cos\left(\frac{\pi r(2p-1)}{q}\right) \frac{\mathrm{d}^{2n-1}}{\mathrm{d}\xi^{2n-1}} \csc(\pi\xi)\bigg|_{\xi=r/q} = (-1)^n \frac{(2\pi)^{2n-1}}{n}\left[q^{2n} B_{2n}\left(\frac{2p-1}{2q}\right) - B_{2n}\left(\frac{1}{2}\right)\right] \quad (p = 1, \ldots, q) \tag{2.9*a}$$

and

$$\sum_{r=1}^{q-1} \sin\left(\frac{\pi r(2p-1)}{q}\right) \frac{\mathrm{d}^{2n}}{\mathrm{d}\xi^{2n}} \csc(\pi\xi)\bigg|_{\xi=r/q} = (-1)^{n-1} \frac{2(2\pi)^{2n}}{2n+1} q^{2n+1} B_{2n+1}\left(\frac{2p-1}{2q}\right) \quad (p = 1, \ldots, q-1) \tag{2.9*b}$$

are the previously unknown companions to the Wang sums (see Remark 2).

3. Proof of the results

Proof of Theorems 1 and 2. Our proofs of Theorems 1 and 2 are based on the following two discrete Fourier transform pairs valid for any complex s with $s \neq 1$.

The first pair is given by

$$\zeta\left(s, \frac{p}{q}\right) = \frac{1}{q}\sum_{r=1}^{q} q^s \ell_s\left(\frac{r}{q}\right) \mathrm{e}^{-\frac{2\pi \mathrm{i} rp}{q}} \quad (p = 1, \ldots, q) \tag{3.1}$$

and

$$\ell_s\left(\frac{r}{q}\right) = \frac{1}{q^s}\sum_{p=1}^{q} \zeta\left(s, \frac{p}{q}\right) \mathrm{e}^{\frac{2\pi \mathrm{i} pr}{q}} \quad (r = 1, \ldots, q), \tag{3.2}$$

where $\zeta(s, a)$ and $\ell_s(\xi)$ are the Hurwitz and Lerch zeta functions, while the Legendre chi function $\chi_s(z)$ and $\zeta(s, a)$ constitute the second pair

$$\zeta\left(s, \frac{2p-1}{2q}\right) = \frac{1}{q}\sum_{r=1}^{q} (2q)^s \chi_s\left(\mathrm{e}^{\frac{\pi \mathrm{i} r}{q}}\right) \mathrm{e}^{-\frac{2\pi \mathrm{i}(2r-1)p}{q}} \quad (p = 1, \ldots, q) \tag{3.3}$$

and

$$\chi_s\left(\mathrm{e}^{\frac{\pi \mathrm{i} r}{q}}\right) = \frac{1}{(2q)^s}\sum_{p=1}^{q} \zeta\left(s, \frac{2p-1}{2q}\right) \mathrm{e}^{\frac{2\pi \mathrm{i}(2p-1)r}{q}} \quad (r = 1, \ldots, q). \tag{3.4}$$

We first show that (3.1)–(3.4) holds true for the case when $\Re(s) > 1$. Indeed, for $\Re(s) > 1$, from (1.3) we obtain

$$\ell_s\left(\frac{r}{q}\right) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i}(k+1)p/q}}{(k+1)^s} = \sum_{r=0}^{q-1}\sum_{k=0}^{\infty} \frac{\mathrm{e}^{2\pi \mathrm{i} kp} \mathrm{e}^{2\pi \mathrm{i}(r+1)p/q}}{q^s(k + (r+1)/q)^s}$$

so that, in view of the definition of the Hurwitz zeta function in (1.1), we have (3.2). Similarly, when $\Re(s) > 1$, the formula (3.4) follows immediately from (1.3). Next, we establish the formulae (3.1) and (3.3) by employing the Fourier inversion theorem.

Second we shall show that the above-given formulae remain valid $\Re(s) \leq 1$, $s \neq 1$. To do so, observe that (3.1)–(3.4) may be extended by analytic continuation on s as far as possible. It is well known that the Hurwitz and Riemann zeta functions, $\zeta(s, a)$ and $\zeta(s)$, are meromorphic in $s \in \mathbb{C}$, with a sole simple pole at $s = 1$. If ξ is not an integer, $\ell_s(\xi)$ is an entire function in $s \in \mathbb{C}$, and for an integer ξ it reduces to $\zeta(s)$. Similarly, the Legendre chi function $\chi_s(z)$ is meromorphic with simple pole at $s = 1$. We thus conclude that the formulae (3.1)–(3.4) hold true for any complex s, $s \neq 1$.

Finally, in view of the known relation [5, p. 85, Eq. (17)]

$$\zeta(1-n, a) = -\frac{1}{n} B_n(a) \quad (n \in \mathbb{N}), \tag{3.5}$$

the proposed formulae (2.1) and (2.2) as well as (2.5) and (2.6) follow upon noting that (3.1)–(3.4) are valid for $s = 1-n$ ($n \in \mathbb{N}$). □

Proof of Corollaries 1A, 1B and 2. First, note that (2.1) and (2.7) could be rewritten in the form given by (2.1*) and (2.7*).

Next, we shall show that

$$\ell_0(\xi) = -\frac{1}{2} + \frac{\mathrm{i}}{2}\cot(\pi\xi), \qquad \ell_{1-n}(\xi) = \frac{\mathrm{i}}{2(2\pi\mathrm{i})^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\cot(\pi\xi) \quad (\xi\in\mathbb{R}\setminus\mathbb{Z};\ n\in\mathbb{N}\setminus\{1\}) \tag{3.6}$$

and

$$\chi_{1-n}(\mathrm{e}^{\pi\mathrm{i}\xi}) = \frac{\mathrm{i}}{2(\pi\mathrm{i})^{n-1}}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\xi^{n-1}}\csc(\pi\xi) \quad (\xi\in\mathbb{R}\setminus\mathbb{Z};\ n\in\mathbb{N}). \tag{3.7}$$

To prove (3.6) note that

$$\frac{\partial}{\partial\xi}\ell_s(\xi) = 2\pi\mathrm{i}\,\ell_{s-1}(\xi), \tag{3.8}$$

which, in turn, follows from (1.2) for $\Re(s) > 2$ and by analytic continuation for all s. The definition in (1.2) also yields $\ell_1(\xi) = -\log(1-\mathrm{e}^{2\pi\mathrm{i}\xi})(\xi\in\mathbb{R}\setminus\mathbb{Z})$ and we from this obtain $\ell_0(\xi)$ by (3.8). Using (3.8) repeatedly with initial value $\ell_0(\xi)$ leads to the expression in (3.6) for $\ell_{1-n}(\xi)$.

Likewise, we have (3.7) by making use of

$$\frac{\partial}{\partial\xi}\chi_s(\mathrm{e}^{\pi\mathrm{i}\xi}) = \pi\mathrm{i}\,\chi_{s-1}(\mathrm{e}^{\pi\mathrm{i}\xi})$$

and

$$\chi_0(\mathrm{e}^{\pi\mathrm{i}\xi}) = \frac{\mathrm{i}}{2}\csc(\pi\xi) \quad (\xi\in\mathbb{R}\setminus\mathbb{Z}).$$

Lastly, upon substituting the obtained formula for $\ell_{1-n}(\xi)$ ($\ell_0(\xi)$) given by (3.6) into (2.1*) and (2.2) we arrive at the proposed assertions of Corollary 1A (Corollary 1B). In similar manner, by (3.7), (2.7*) and (2.8), we prove Corollary 2. □

4. Additional results

We begin this section by listing several first values of $\ell_{-n}(\xi)$.

Examples 1. In view of (3.6) we have (*cf.* [2, p. 227]):

$$\begin{aligned}
\ell_{-1}(\xi) &= -\frac{1}{4}\left[1+\cot^2(\pi\xi)\right],\\
\ell_{-2}(\xi) &= -\frac{\mathrm{i}}{8}\left[2\cot(\pi\xi)+2\cot^3(\pi\xi)\right],\\
\ell_{-3}(\xi) &= \frac{1}{16}\left[2+8\cot^2(\pi\xi)+6\cot^4(\pi\xi)\right],\\
\ell_{-4}(\xi) &= \frac{\mathrm{i}}{32}\left[16\cot(\pi\xi)+40\cot^3(\pi\xi)+24\cot^5(\pi\xi)\right],\\
\ell_{-5}(\xi) &= -\frac{1}{64}\left[16+136\cot^2(\pi\xi)+240\cot^4(\pi\xi)+120\cot^6(\pi\xi)\right],\\
\ell_{-6}(\xi) &= -\frac{\mathrm{i}}{128}\left[272\cot(\pi\xi)+1232\cot^3(\pi\xi)+1680\cot^5(\pi\xi)+720\cot^7(\pi\xi)\right],\\
\ell_{-7}(\xi) &= -\frac{1}{256}\left[272-3968\cot^2(\pi\xi)-12\,096\cot^4(\pi\xi)-13\,440\cot^6(\pi\xi)-5040\cot^8(\pi\xi)\right],\\
\ell_{-8}(\xi) &= -\frac{\mathrm{i}}{512}\left[7936\cot(\pi\xi)+56\,320\cot^3(\pi\xi)+129\,024\cot^5(\pi\xi)+120\,960\cot^7(\pi\xi)+40\,320\cot^9(\pi\xi)\right].
\end{aligned}$$

Remark 5 (*Williams–Zhang Sums*). It is easily seen that, upon examining Examples 1, the left-hand side of the Wang sums (2.1*) with values of $\ell_{1-n}(\xi)$, $n \geq 2$, from Examples 1 takes two different forms depending on parity of n: in the case of even n it becomes a linear combination of $C_{2k}(q,p)(k = 0,\ldots,\lfloor n/2\rfloor)$, while for odd n, $n \geq 3$, it is a linear combination of $S_{2k+1}(q,p)(k = 0,\ldots,\lfloor n/2\rfloor)$, where $C_{2k}(q,p)$ and $S_{2k+1}(q,p)$ are the following sums

$$C_{2k}(q,p) = \sum_{r=1}^{q-1}\cos\left(\frac{2r\pi p}{q}\right)\cot^{2k}\left(\frac{r\pi}{q}\right) \tag{4.1}$$

and

$$S_{2k+1}(q,p)=\sum_{r=1}^{q-1}\sin\left(\frac{2r\pi p}{q}\right)\cot^{2k+1}\left(\frac{r\pi}{q}\right). \tag{4.2}$$

Williams and Zhang ([4]; see also [7]) generalized the Eisenstein sum (2.3*) by summing the trigonometric sums in (4.1) and (4.2), $C_{2k}(q,p)$, $k\geq 1$, and $S_{2k+1}(q,p)$, $k\geq 0$. It follows from this analysis that the Williams–Zhang sums can be recovered from the Wang sums (2.1*) in conjunction with (3.6). All that is needed is to know that $C_0(q,p)=-1$ and that $S_1(q,p)$ is the Eisenstein sum given in (2.3*). Thus, we obtain:

$$C_2(q,p)=\frac{2}{3}+2q^2B_2\left(\frac{p}{q}\right),$$
$$S_3(q,p)=2qB_1\left(\frac{p}{q}\right)+\frac{4}{3}q^3B_3\left(\frac{p}{q}\right),$$
$$C_4(q,p)=-\frac{26}{45}-\frac{8}{3}q^2B_2\left(\frac{p}{q}\right)-\frac{2}{3}q^4B_4\left(\frac{p}{q}\right),$$
$$S_5(q,p)=-2qB_1\left(\frac{p}{q}\right)-\frac{20}{9}q^3B_3\left(\frac{p}{q}\right)-\frac{4}{15}q^5B_5\left(\frac{p}{q}\right),$$
$$C_6(q,p)=\frac{502}{945}+\frac{46}{15}q^2B_2\left(\frac{p}{q}\right)+\frac{4}{3}q^4B_4\left(\frac{p}{q}\right)+\frac{4}{45}q^6B_6\left(\frac{p}{q}\right),$$
$$S_7(q,p)=2qB_1\left(\frac{p}{q}\right)+\frac{392}{135}q^3B_3\left(\frac{p}{q}\right)+\frac{28}{45}q^5B_5\left(\frac{p}{q}\right)+\frac{8}{315}q^7B_7\left(\frac{p}{q}\right),$$
$$C_8(q,p)=-\frac{7102}{14\,175}-\frac{352}{105}q^2B_2\left(\frac{p}{q}\right)-\frac{88}{45}q^4B_4\left(\frac{p}{q}\right)-\frac{32}{135}q^6B_6\left(\frac{p}{q}\right)-\frac{2}{315}q^8B_8\left(\frac{p}{q}\right).$$

Examples 2. In view of (3.7) we have:

$$\chi_{-1}(e^{\pi i\xi})=-\frac{1}{2}\cot(\pi\xi)\csc(\pi\xi),$$
$$\chi_{-2}(e^{\pi i\xi})=-\frac{i}{2}\left[\csc(\pi\xi)+2\cot^2(\pi\xi)\csc(\pi\xi)\right],$$
$$\chi_{-3}(e^{\pi i\xi})=\frac{1}{2}\left[5\cot(\pi\xi)\csc(\pi\xi)+6\cot^3(\pi\xi)\csc(\pi\xi)\right],$$
$$\chi_{-4}(e^{\pi i\xi})=\frac{i}{2}\left[5\csc(\pi\xi)+28\cot^2(\pi\xi)\csc(\pi\xi)+24\cot^4(\pi\xi)\csc(\pi\xi)\right],$$
$$\chi_{-5}(e^{\pi i\xi})=-\frac{1}{2}\left[61\cot(\pi\xi)\csc(\pi\xi)+180\cot^3(\pi\xi)\csc(\pi\xi)+120\cot(\pi\xi)^5\csc(\pi\xi)\right],$$
$$\chi_{-6}(e^{\pi i\xi})=-\frac{i}{2}\left[61\csc(\pi\xi)+662\cot^2(\pi\xi)\csc(\pi\xi)+1320\cot(\pi\xi)^4\csc(\pi\xi)+720\cot^6(\pi\xi)\csc(\pi\xi)\right].$$

Remark 6 (*New Sums*). By analysis analogous to that in Remark 5, by making use of (2.7*) and (3.7), we arrive at the following (presumably) new summation formulae

$$\mathcal{S}_0(q,p)=-2qB_1\left(\frac{2p-1}{2q}\right),$$
$$\mathcal{C}_1(q,p)=-2B_2\left(\frac{1}{2}\right)+2q^2B_2\left(\frac{2p-1}{2q}\right),$$
$$\mathcal{S}_2(q,p)=qB_1\left(\frac{2p-1}{2q}\right)+\frac{4}{3}q^3B_3\left(\frac{2p-1}{2q}\right),$$
$$\mathcal{C}_3(q,p)=\frac{5}{3}B_2\left(\frac{1}{2}\right)+\frac{2}{3}B_4\left(\frac{1}{2}\right)-\frac{5}{3}q^2B_2\left(\frac{2p-1}{2q}\right)-\frac{2}{3}q^4B_4\left(\frac{2p-1}{2q}\right),$$
$$\mathcal{S}_4(q,p)=-\frac{3}{4}qB_1\left(\frac{2p-1}{2q}\right)-\frac{14}{9}q^3B_3\left(\frac{2p-1}{2q}\right)-\frac{4}{15}q^5B_5\left(\frac{2p-1}{2q}\right)$$

$$\mathcal{C}_5(q,p) = -\frac{89}{60}B_2\left(\frac{1}{2}\right) - B_4\left(\frac{1}{2}\right) - \frac{4}{45}B_6\left(\frac{1}{2}\right) + \frac{89}{60}q^2B_2\left(\frac{2p-1}{2q}\right) + q^4B_4\left(\frac{2p-1}{2q}\right) + \frac{4}{45}q^6B_6\left(\frac{2p-1}{2q}\right),$$

$$\mathcal{S}_6(q,p) = \frac{5}{8}qB_1\left(\frac{2p-1}{2q}\right) + \frac{439}{270}q^3B_3\left(\frac{2p-1}{2q}\right) + \frac{22}{45}q^5B_5\left(\frac{2p-1}{2q}\right) + \frac{8}{315}q^7B_7\left(\frac{2p-1}{2q}\right),$$

where

$$\mathcal{S}_{2k}(q,p) = \sum_{r=1}^{q-1}\sin\left(\frac{r\pi(2p-1)}{q}\right)\cot^{2k}\left(\frac{r\pi}{q}\right)\csc\left(\frac{r\pi}{q}\right) \tag{4.3}$$

and

$$\mathcal{C}_{2k+1}(q,p) = \sum_{r=1}^{q-1}\cos\left(\frac{r\pi(2p-1)}{q}\right)\cot^{2k+1}\left(\frac{r\pi}{q}\right)\csc\left(\frac{r\pi}{q}\right). \tag{4.4}$$

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