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The Haruki–Rassias and related integral representations of the Bernoulli and Euler polynomials

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Abstract

Haruki and Rassias [H. Haruki, T.M. Rassias, New integral representations for Bernoulli and Euler polynomials, J. Math. Anal. Appl. 175 (1993) 81–90] found the integral representations of the classical Bernoulli and Euler polynomials and proved them by making use of the properties of certain functional equation. In this sequel, we rederive, in a completely different way, the results of Haruki and Rassias and deduce related and new integral representations. Our proofs are quite simple and remarkably elementary.

Keywords: Bernoulli polynomials; Euler polynomials; Integral representations

1. Introduction

In the usual notations, let $B_n(x)$ and $E_n(x)$ denote, respectively, the classical Bernoulli and Euler polynomials of degree n in x, defined for each nonnegative integer n by means of their exponential generating functions [1, p. 804, Eq. 23.1.1]

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\big(|t|<2\pi\big) \tag{1.1}$$

and

$$\frac{2te^{tx}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad\big(|t|<\pi\big). \tag{1.2}$$

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Haruki and Rassias [2] found the integral representations of $B_n(x)$ and $E_n(x)$ and they, in the proof, used extensively a certain functional equation and a number of the properties of $B_n(x)$ and $E_n(x)$.

In this sequel, by making use of elementary arguments, we will derive the results of Haruki and Rassias and find several related, presumably new, integral representations of the classical Bernoulli and Euler polynomials of degree n in x. Our proofs are quite simple and remarkably elementary.

2. Statement of the results

Haruki and Rassias [2, Theorem] obtained the four integral representations which are given below by our Theorem 1(a) and (d) and Corollary 1(a) and (d). We have been unable to find the remaining four integrals in the literature.

Theorem 1. *Let $n \in \mathbb{N}$. If x is a real number, $0 \leqslant x \leqslant 1$, then*

$$\text{(a)}\quad B_{2n-1}(x) = (-1)^n \frac{2(2n-1)}{(2\pi)^{2n-1}} \int_0^1 (\log t)^{2n-2} \frac{\sin(2\pi x)}{t^2 - 2t\cos(2\pi x) + 1}\, dt,$$

$$\text{(b)}\quad B_{2n}(x) = (-1)^n \frac{2(2n)}{(2\pi)^{2n}} \int_0^1 (\log t)^{2n-1} \frac{\cos(2\pi x) - t}{t^2 - 2t\cos(2\pi x) + 1}\, dt,$$

$$\text{(c)}\quad E_{2n-1}(x) = (-1)^n \frac{4}{\pi^{2n}} \int_0^1 (\log t)^{2n-1} \frac{(t^2-1)\cos(\pi x)}{t^4 - 2t^2\cos(2\pi x) + 1}\, dt,$$

$$\text{(d)}\quad E_{2n}(x) = (-1)^n \frac{4}{\pi^{2n+1}} \int_0^1 (\log t)^{2n} \frac{(t^2+1)\sin(\pi x)}{t^4 - 2t^2\cos(2\pi x) + 1}\, dt.$$

Corollary 1. *Let $n \in \mathbb{N}$. If x is a real number, $0 \leqslant x \leqslant 1$, then*

$$\text{(a)}\quad B_{2n}(x) = (-1)^n \frac{2n(2n-1)}{(2\pi)^{2n}} \int_0^1 (\log t)^{2n-2} \frac{1}{t} \log\bigl(t^2 - 2t\cos(2\pi x) + 1\bigr)\, dt,$$

$$\text{(b)}\quad B_{2n+1}(x) = (-1)^n \frac{4n(2n+1)}{(2\pi)^{2n+1}} \int_0^1 (\log t)^{2n-1} \frac{1}{t} \arctan\left(\frac{t\sin(2\pi x)}{1 - t\cos(2\pi x)}\right) dt,$$

$$\text{(c)}\quad E_{2n}(x) = (-1)^n \frac{4n}{\pi^{2n+1}} \int_0^1 (\log t)^{2n-1} \frac{1}{t} \arctan\left(\frac{2t\sin(\pi x)}{t^2 - 1}\right) dt,$$

$$\text{(d)}\quad E_{2n+1}(x) = (-1)^n \frac{(2n+1)}{\pi^{2n+2}} \int_0^1 (\log t)^{2n} \frac{1}{t} \log\left(\frac{t^2 - 2t\cos(\pi x) + 1}{t^2 + 2t\cos(\pi x) + 1}\right) dt.$$

3. Proof of the results

In order to prove the above given representations we use several quite simple series expansions and integrals, which we, for the sake of ready reference, list as lemmas. Note that all of these expansions and integrals can be found in the standard reference books, for instance in [3].

Some basic properties of the Bernoulli and the Euler polynomials are also required and, for more details, we refer the reader to the books [1] and [4].

Lemma 1. *Let x be a real number. If t is a complex number,* $|t|<1$*, then*

$$\text{(a)}\quad \frac{t\sin(2\pi x)}{1-2t\cos(2\pi x)+t^2}=\sum_{k=1}^{\infty}t^k\sin(2k\pi x),$$

$$\text{(b)}\quad \frac{t(\cos(2\pi x)-t)}{1-2t\cos(2\pi x)+t^2}=\sum_{k=1}^{\infty}t^k\cos(k2\pi x),$$

$$\text{(c)}\quad \frac{t(1+t^2)\sin(\pi x)}{1-2t^2\cos(2\pi x)+t^4}=\sum_{k=0}^{\infty}t^{2k+1}\sin(2k+1)\pi x,$$

$$\text{(d)}\quad \frac{t(1-t^2)\cos(\pi x)}{1-2t^2\cos(2\pi x)+t^4}=\sum_{k=0}^{\infty}t^{2k+1}\cos(2k+1)\pi x.$$

Lemma 2. *We have*

$$\text{(a)}\quad \int\frac{t\sin x}{1-2t\cos x+t^2}\,dx=\frac{1}{2}\log\bigl(1-2t\cos x+t^2\bigr)+C,$$

$$\text{(b)}\quad \int\frac{t(\cos x-t)}{1-2t\cos x+t^2}\,dx=\arctan\frac{t\sin x}{1-t\cos x}+C,$$

$$\text{(c)}\quad \int\frac{2t(1+t^2)\sin x}{1-2t^2\cos(2x)+t^4}\,dx=\frac{1}{2}\log\frac{1-2t\cos x+t^2}{1+2t\cos x+t^2}+C,$$

$$\text{(d)}\quad \int\frac{2t(1-t^2)\cos x}{1-2t^2\cos(2x)+t^4}\,dx=\arctan\frac{2t\sin x}{1-t^2}+C.$$

Proof of Theorem 1. The proof is simple and rests on Lemma 1 and the well-known Fourier expansions of the Bernoulli and Euler polynomials.

We first establish the part (a). By Lemma 1(a) we have

$$\frac{\sin(2\pi x)}{1-2t\cos(2\pi x)+t^2}=\sum_{k=1}^{\infty}t^{k-1}\sin(2k\pi x)$$

so that

$$\int_0^1(\log t)^{2n-2}\frac{\sin(2\pi x)}{t^2-2t\cos(2\pi x)+1}\,dt=\sum_{k=1}^{\infty}\sin(2k\pi x)\left(\int_0^1 t^{k-1}(\log t)^{2n-2}\,dt\right)$$

$$=\sum_{k=1}^{\infty}\sin(2k\pi x)\frac{(2n-2)!}{k^{2n-1}}=(-1)^n\frac{(2\pi)^{2n-1}}{2(2n-1)}B_{2n-1}(x)$$

which is the sought result. Here, we first evaluate the needed integral by [3, p. 488, Eq. 2.6.3.2]

$$\int_0^1 t^{\alpha-1}\log^n t\,dt = (-1)^n \frac{n!}{\alpha^{n+1}} \quad \bigl(n \in \mathbb{N};\ \Re(\alpha) > 0\bigr), \tag{3.1}$$

and then employ the expansion [1, p. 805, Eq. 23.1.17]

$$\sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n-1}} = (-1)^n \frac{(2\pi)^{2n-1}}{2(2n-1)!} B_{2n-1}(x)$$

where $0 \leqslant x \leqslant 1$ for $n = 2, 3, \ldots$, and $0 < x < 1$ for $n = 1$. Note that inverting the order of summation and integration is justified by absolute convergence of the series involved.

The parts (b), (c) and (d) are proved in precisely the same way, respectively, starting from Lemma 1(b), (c) and (d) and making use of the integral (3.1) and the Fourier expansions of $B_{2n}(x)$, $E_{2n-1}(x)$ and $E_{2n}(x)$ (see, for instance, [1, p. 805, Eqs. 23.1.17 and 23.1.18] and [4, p. 65, Eqs. 55–57]).

This completes our proof. □

Proof of Corollary 1. All integral formulas given by Corollary 1 are deduced by definite integration of both sides of the corresponding formulas in Theorem 1 with respect to x. The details required for the integration of the right-hand sides are summarized in Lemma 2, while the following formulas (cf. [1, p. 805, Eq. 23.1.11] and [4, p. 60, Eq. 14 and p. 64, Eq. 46])

$$\int_0^z \begin{Bmatrix} B_n(x) \\ E_n(x) \end{Bmatrix} dx = \frac{1}{n+1} \begin{Bmatrix} B_{n+1}(z) - B_{n+1}(0) \\ E_{n+1}(z) - E_{n+1}(0) \end{Bmatrix} \quad (n \in \mathbb{N}) \tag{3.2}$$

where

$$B_n(0) = B_n \quad \text{and} \quad E_n(0) = \frac{2}{n+1}\bigl(1 - 2^{n+1}\bigr)B_{n+1},$$

B_n being the nth Bernoulli number, are used for the integration of the left-hand sides.

Corollary 1(b) and (c) follow at once upon integration of the expressions in Theorem 1(b) and (c) since $B_{2n+1} = 0$ ($n \in \mathbb{N}$). In the proof of the parts (a) and (d) we also need the integrals [3, p. 526, Eq. 2.6.19.4]

$$\int_0^1 \begin{Bmatrix} \log(1+t) \\ \log(1-t) \end{Bmatrix} \frac{\log^n t}{t}\,dt = (-1)^{n-1} n!\,\zeta(n+2) \begin{Bmatrix} (2^{-(n+1)} - 1) \\ 1 \end{Bmatrix} \quad (n \in \mathbb{N})$$

and the celebrated Euler relation between the even-indexed Bernoulli numbers and the values of the Riemann zeta function (denoted by $\zeta(s)$) at the even integers [4, p. 98, Eq. 18]

$$B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) \quad (n \in \mathbb{N}_0).$$

The proof is now complete. □

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