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Some discrete Fourier transform pairs associated with the Lipschitz–Lerch Zeta function

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ABSTRACT

It is shown that there exists a companion formula to Srivastava's formula for the Lipschitz–Lerch Zeta function [see H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc. 129 (2000) 77–84] and that together these two results form a discrete Fourier transform pair. This Fourier transform pair makes it possible for other (known or new) results involving the values of various Zeta functions at rational arguments to be easily recovered or deduced in a more general context and in a remarkably unified manner.

1. Introduction and definitions

Srivastava's formula [1, p. 81, Eq. (3.9)] reproduced here in (2.2) below, which provides a relationship between the values of the Lipschitz–Lerch and Hurwitz Zeta functions, has been used in several recent papers (see [2, p. 298, Eq. (39)], [3, p. 821, Eq. (23)] and [4, p. 806, Eq. (18)]). In this note we obtain its companion formula and show that together these two results would form a discrete Fourier transform (DFT) pair. For more details regarding the discrete Fourier transforms, the reader is referred to such standard text on the subject as the book by Weaver [5].

A general Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ defined by [6, p. 121 *et seq.*]

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- := \{0,-1,-2,-3,\ldots\}) \\ (s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1) \tag{1.1}$$

contains, as its *special* cases, not only the Lipschitz–Lerch Zeta function [6, p. 122, Eq. 2.5(11)]:

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi(e^{2\pi i\xi},s,a) \\ \big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \Re(s)>0 \text{ when } \xi \in \mathbb{R}\setminus\mathbb{Z};\ \Re(s)>1 \text{ when } \xi \in \mathbb{Z}\big) \tag{1.2}$$

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and the Hurwitz (or generalized) and the Riemann Zeta functions:

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a) \quad \text{and} \quad \zeta(s) = \Phi(1,s,1), \tag{1.3}$$

but also other functions such as the Lerch Zeta function [6, p. 122, Eq. 2.5(11)]:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i \xi}}{n^s} = e^{2\pi i \xi} \Phi(e^{2\pi i \xi}, s, 1) \quad (\xi \in \mathbb{R};\ \Re(s) > 1) \tag{1.4}$$

and the Legendre Chi function $\chi_s(z)$ (see, for instance, [7,8]):

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s} = \frac{1}{2^s} z \Phi\left(z^2, s, \frac{1}{2}\right) \quad (|z| \leqq 1;\ \Re(s) > 1)\,. \tag{1.5}$$

Finally, the classical Bernoulli polynomials $B_n(x)$ and the classical Bernoulli numbers B_n are defined by (see, for details, [6, p. 61 *et seq.*]; see also a recent work [9]):

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi) \quad \text{and} \quad B_n := B_n(0)$$
$$(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.6}$$

2. The main results and their proofs

We begin by observing that, in what follows, we set an empty sum to be zero and it is assumed that p, r and q are positive integers. Our main results in this section are stated and proved as follows. As indicated above, (2.2) is Srivastava's formula [1], while (2.1), (2.3) and (2.4) are presumably new.

Theorem. *Suppose that s and a are complex numbers, $s \neq 1$ and $a \notin \mathbb{Z}_0^-$. Then $\zeta(s, a)$ and $\phi(\xi, a, s)$ form the following* DFT *pair:*

$$\zeta\left(s, \frac{p+a-1}{q}\right) = \frac{1}{q} \sum_{r=1}^{q} q^s \phi\left(\frac{r}{q}, a, s\right) \exp\left(-\frac{2\pi i (r-1)p}{q}\right)$$
$$(p = 1, \ldots, q) \tag{2.1}$$

and

$$\phi\left(\frac{r}{q}, a, s\right) = \frac{1}{q^s} \sum_{p=1}^{q} \zeta\left(s, \frac{p+a-1}{q}\right) \exp\left(\frac{2\pi i (p-1)r}{q}\right)$$
$$(r = 1, \ldots, q). \tag{2.2}$$

Proof. Assume that $\Re(s) > 1$. We first note that Srivastava [1] gave a simple and elegant proof of (2.2) (see, for details, [1, p. 81]). Our proof of (2.1) requires each of the following results:

(a) *Simpson's Series Multisection Formula* (see, for instance, [10, p. 131]). Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^k$$

and let q be fixed. Then, for any p $(1 \leqq p \leqq q)$, we have

$$q \sum_{k=0}^{\infty} a_{p+qk} z^{p+qk} = \sum_{s=1}^{q} \omega^{-sp} f\left(\omega^s z\right) \quad \left(\omega = \exp\left(\frac{2\pi i}{q}\right)\right). \tag{2.3}$$

(b) *Abel's Theorem* (see [11, p. 148]). Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^k.$$

If the series

$$\sum_{k=1}^{\infty} a_k$$

converges, then

$$\lim_{z\to 1-}\{f(z)\} = \sum_{k=1}^{\infty} a_k. \tag{2.4}$$

Now, the assertion (2.1) of the Theorem follows immediately upon making use of Simpson's Series Multisection Formula (2.3) and Abel's Theorem (2.4) on the defining series (1.1) for $\Phi(z, s, a)$.

We next show that the relations in (2.1) and (2.2) form a discrete Fourier transform pair. Indeed, upon substituting from (2.1) into (2.2) and using the corresponding orthogonality property, we have

$$\begin{aligned}\phi\left(\frac{r}{q}, a, s\right) &= \frac{1}{q}\sum_{p=1}^{q}\sum_{r=1}^{q}\phi\left(\frac{r}{q}, a, s\right)\exp\left(-\frac{2\pi \mathrm{i} r}{q}\right)\cdot\exp\left(\frac{2\pi \mathrm{i} p}{q}\right)\\ &= \phi\left(\frac{r}{q}, a, s\right) \quad (r = 1, \ldots, q).\end{aligned} \tag{2.5}$$

Thus the proposed transform relations (2.1) and (2.2) are established for $\Re(s) > 1$.

Lastly, it is clear that the above formulas (2.1) and (2.2) may be extended by applying the principle of analytic continuation on s as far as possible. It is well known that $\zeta(s, a)$ is a meromorphic function in $s \in \mathbb{C}$, with a single *simple* pole as $s = 1$. If ξ is not an integer, $\phi(\xi, a, z)$ is an entire function in $s \in \mathbb{C}$. Moreover, for an integer ξ, the entire function $\phi(\xi, a, z)$ reduces to $\zeta(s, a)$. In other words, the formulas (2.1) and (2.2) are valid for any complex s $(s \neq 1)$. □

Corollary. *Suppose that n is a positive integer and that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$. Then, in terms of $B_n(x)$ and $\phi(\xi, a, s)$, the following* DFT *pair holds true:*

$$-\frac{1}{n}B_n\left(\frac{p+a-1}{q}\right) = \frac{1}{q}\sum_{r=1}^{q} q^{1-n}\phi\left(\frac{r}{q}, a, 1-n\right)\exp\left(-\frac{2\pi \mathrm{i}(r-1)p}{q}\right)$$
$$(p = 1, \ldots, q), \tag{2.6}$$

and

$$\phi\left(\frac{r}{q}, a, 1-n\right) = -\frac{1}{nq^{1-n}}\sum_{p=1}^{q} B_n\left(\frac{p+a-1}{q}\right)\exp\left(\frac{2\pi \mathrm{i}(p-1)r}{q}\right)$$
$$(r = 1, \ldots, q). \tag{2.7}$$

Proof. The above Corollary follows from our Theorem in conjunction with the following familiar relationship [6, p. 85, Eq. (17)]:

$$\zeta(1-n, a) = -\frac{B_n(a)}{n} \quad (n \in \mathbb{N}), \tag{2.8}$$

where $B_n(x)$ are the Bernoulli polynomials defined by (1.6). □

Remark. Yet another immediate consequence of the Theorem is a pair of new transform relations which could be deduced by simultaneous use of (2.8) and Apostol's formula (*cf.*, *e.g.*, [2, p. 299, Eq. (46)]):

$$\phi(\xi, a, 1-n) = -\frac{\mathcal{B}_n(a; \mathrm{e}^{2\pi \mathrm{i}\xi})}{n} \quad (n \in \mathbb{N}), \tag{2.9}$$

$\mathcal{B}_n(x; \lambda)$ being the Apostol–Bernoulli polynomials (see, for instance, [2, p. 291, Eq. (5)]).

3. A set of interesting special cases

We note that the *main* results in two papers by Cvijović and Klinowski (see [12,8]), which were also proved to be DFT pairs, are special cases of the above Theorem. The first result of Cvijović and Klinowski [12, p. 48, Theorem]:

$$\zeta\left(s, \frac{p}{q}\right) = \frac{1}{q}\sum_{r=1}^{q} q^s \ell_s\left(\frac{r}{q}\right)\exp\left(-\frac{2\pi \imath rp}{q}\right) \quad (p = 1, \ldots, q) \tag{3.1}$$

and

$$\ell_s\left(\frac{r}{q}\right) = \frac{1}{q^s}\sum_{p=1}^{q}\zeta\left(s, \frac{p}{q}\right)\exp\left(\frac{2\pi \imath pr}{q}\right) \quad (r = 1, \ldots, q) \tag{3.2}$$

can be recovered by setting $a = 1$ in (2.1) and (2.2) and making use of the definition (1.4) of the Lerch Zeta function $\ell_s(\xi)$. The second result of Cvijović and Klinowski [8, p. 1625, Theorem] would follow similarly by setting $a = \frac{1}{2}$ and making use of the definition (1.5) of the Legendre Chi function $\chi_s(z)$. For various more than two dozen formulas, most of which being previously unknown, which were established as a consequence of these pairs of results, the interested reader is referred to the aforecited papers by Cvijović and Klinowski (see [12,8]).

Clearly, in the same way as detailed above, two more pairs of results can be obtained from the above Corollary, but we choose only to record, in the following Example, the case when $a = 1$. The formula (3.3) below is essentially the same as the result which was derived, by using markedly different arguments, by Wang [13, p. 12, Theorem D]. In addition, many new formulas involving the values of various Zeta functions at rational arguments can be easily deduced by appealing to the relations mentioned in the above Remark.

Example. If $n \in \mathbb{N}$, then it is easily seen that

$$-\frac{1}{n}B_n\left(\frac{p}{q}\right) = \frac{1}{q}\sum_{r=1}^{q} q^{1-n}\ell_{1-n}\left(\frac{r}{q}\right)\exp\left(-\frac{2\pi\,\mathrm{i}rp}{q}\right) \quad (p = 1, \ldots, q) \tag{3.3}$$

and

$$\ell_{1-n}\left(\frac{r}{q}\right) = -\frac{1}{nq^{1-n}}\sum_{p=1}^{q} B_n\left(\frac{p}{q}\right)\exp\left(\frac{2\pi\,\mathrm{i}pr}{q}\right) \quad (r = 1, \ldots, q). \tag{3.4}$$

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