

A brief introduction to black holes

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Abstract:

Introduction/purpose: Starting from general relativity, black hole generation and effects are investigated.

Method: Einstein's equation and its Schwarzschild solution are employed to study black holes. Quantum mechanics is used to obtain Hawking's radiation.

Results: Black holes are actually not completely black - they radiate energy during their lifetime.

Conclusions: Black holes could evaporate and this effect is observable if their mass is sufficiently small. Their entropy scales differently with respect to their mass from that of other objects in thermodynamics.

Key words: general relativity, black Holes, Hawking radiation.

General relativity

Special relativity (SR) is a theory that applies to systems moving at a constant velocity. The speed of light c is constant for all observers; space and time are separate and independent dimensions. The metric that defines the distance ds^2 between two points of the four dimensional space is not positive defined:

$$ds^2 = c^2 dt^2 - dx^2 \quad (1)$$

The corresponding metric is given by $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, $\mu, \nu = 1, \dots, 4$. In SR the metric tensor $g_{\mu\nu}$ does not depend on the point, and the space is flat. The Lorentz group is the group of spacetime transformations that preserve (1).

General relativity (GR) applies to systems accelerating or under the influence of gravity, which can bend rays of light. This is the case in the presence of very massive objects such as black holes.

In GR, the distance between two points x^μ, x^ν is usually written as

$$s^2 = g_{\mu\nu}x^\mu x^\nu \quad (2)$$

(we use units for which $c = 1$) where $g_{\mu\nu}$ depends upon the spacetime point; therefore, contrary to SR, spacetime is not flat. Curvilinear coordinates are used, where a covariant vector transforms like

$$A^\mu = \frac{dx^{\mu'}}{dx^\mu} A'^\mu, \quad (3)$$

while to obtain a contravariant vector one lowers indices with $g_{\mu\nu}$: $A_\mu = g_{\mu\nu}A^\nu$.

For a vector A , if A^μ is its value at the point x^μ , at its neighbor point $x^\mu + dx^\mu$ will be $A^\mu + dA^\mu$. The total variation of A will be given by coordinates change and a functional variation of A :

$$dA^\mu + \delta A^\mu, \quad (4)$$

where

$$\delta A^\mu = \Gamma_{\rho\sigma}^\mu A^\rho dx^\sigma \quad (5)$$

$\Gamma_{\rho\sigma}^\mu$ is the Christoffel symbol that describes the metric g , non flat in GR. This leads to an extension to the concept of derivative, the covariant derivative being defined as

$$D_\mu A^\nu = \partial_\mu A^\nu - \Gamma_{\mu\rho}^\nu A^\rho. \quad (6)$$

Observe that this concept is also used in quantum field theories where spacetime is flat in order to describe an "inner" gauge space.

The relation of Γ with the metric g is given by:

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}) \quad (7)$$

The Riemann strength tensor could be expressed by a commutator of covariant derivatives (6)

$$R_{\nu\rho\sigma}^\mu = (\partial_\rho \Gamma_{\sigma\nu}^\mu + \Gamma_{\rho\tau}^\mu \Gamma_{\sigma\nu}^\tau) - (\partial_\sigma \Gamma_{\rho\nu}^\mu + \Gamma_{\sigma\tau}^\mu \Gamma_{\rho\nu}^\tau), \quad (8)$$

and, again, in analogy to it the electromagnetic and chromomagnetic field strength $F_{\mu\nu}$ is written as a commutator of covariant gauge derivatives.

The Ricci tensor is defined by contracting two indices of the Riemann tensor:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\sigma\mu\nu} , \quad (9)$$

and the scalar tensor is obtained from the contraction of the remaining two indices of the Ricci tensor:

$$R = g^{\mu\nu} R_{\mu\nu} . \quad (10)$$

The action of GR (in absence of matter and a cosmological constant) is obtained by the simplest scalar of the theory (the Ricci tensor) integrated over the spacetime volume

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} , \quad (11)$$

g being the determinant of $g^{\mu\nu}$. This action is known as the Einstein–Hilbert action, to name just a few papers on the subject: (Einstein, 1915a,b; Hilbert, 1915).

In order to obtain Einstein’s equation of GR, one has to compute the variation of action (11) which is composed of three terms:

$$\begin{aligned} \delta S = & \int d^4x (\delta\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \int d^4x \sqrt{-g} (\delta g^{\mu\nu}) R_{\mu\nu} + \\ & \int d^4x \sqrt{-g} g^{\mu\nu} \delta(R_{\mu\nu}) = \delta S_1 + \delta S_2 + \delta S_3 \end{aligned} \quad (12)$$

Consider the first term δS_1 . For an invertible matrix M , one has $MM^{-1} = I$. It follows that $\delta(MM^{-1}) = 0 = \delta(M)M^{-1} + M\delta(M^{-1}) = 0$. Therefore, $\delta(M^{-1}) = -M^{-1}\delta(M)M^{-1}$ so one obtains

$$\delta(g^{\rho\sigma}) = -g^{\rho\mu} \delta(g_{\mu\nu}) g^{\nu\sigma} \quad (13)$$

For the second term δS_2 , $\log \det M = \text{Tr} \log M$. Therefore, $\delta(\log \det M) = \delta(\det M)/\det M$, that implies $\delta(\det M) = (\det M)\delta(\log \det M)$, then follows that $(\det M)\delta(\text{Tr} \log M) =$

$(\det M) \operatorname{Tr} \delta(\log M) = (\det M) \operatorname{Tr}(M^{-1} \delta(M))$. We obtain:

$$\delta(\sqrt{-g}) = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta(g_{\mu\nu}) \quad (14)$$

For the third term, δS_3 , there is a lengthy and not very enlightening calculation. From the Palatini identity

$$\delta R_{\mu\sigma\lambda}^{\rho} = D_{\sigma} \delta \Gamma_{\mu\lambda}^{\rho} - D_{\lambda} \delta \Gamma_{\mu\sigma}^{\rho} \quad (15)$$

which stems out when considering locally flat coordinates, that is $\Gamma = 0$ for all indices, and therefore for $\delta R_{\mu\sigma\lambda}^{\rho}$ remain only the parts with derivatives, one obtains for the Ricci tensor

$$\delta R_{\mu\nu} = \delta R_{\mu\sigma\lambda}^{\rho} \cdot \quad (16)$$

The result is

$$\delta S_3 = \int d^4x \sqrt{-g} g^{\mu\nu} (D_{\sigma} \delta \Gamma_{\mu\lambda}^{\rho} - D_{\lambda} \delta \Gamma_{\mu\sigma}^{\rho}), \quad (17)$$

integrating by parts and using the fact that $D_{\rho} g^{\mu\nu} = 0$ the third term is just a surface term, thus irrelevant.

The final result of the Einstein-Hilbert action variation is thus

$$\begin{aligned} \delta S = \int d^4x \sqrt{-g} \left[\left(\frac{1}{2} g^{\mu\nu} R \right) \delta(g_{\mu\nu}) + R_{\sigma\rho} (-g^{\sigma\mu} \delta(g_{\mu\nu}) g^{\nu\rho}) \right] = \\ \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) \delta(g_{\mu\nu}) = 0, \end{aligned} \quad (18)$$

which has to be true for any $\delta(g_{\mu\nu})$. Therefore, Einstein's equation for GR is given by

$$\frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} = 0, \quad (19)$$

for an empty space.

Black holes

Perhaps the first "proto-idea" of a black hole came from Laplace, and independently, from Mitchell ([Michell, 1784](#)). He envisaged a Newtonian gravitational field so strong that even the fastest object could not escape

it. If M is the mass generating this particular gravitational field, and m the object that should escape it, then its energy is given by

$$E = \frac{1}{2}mv^2 - G\frac{mM}{r} \quad (20)$$

where G is the gravitational constant. As the maximal possible speed is c , the object will reach its maximal distance r_s from the massive gravitational field for $E = 0$, and from eq. (20) we obtain

$$r_s = \frac{2GM}{c^2} \quad (21)$$

So even the light cannot leave this massive object for a distance larger than $r > r_s$. Therefore, this particular object cannot emit light at a distance, so it appears to be black. This amazing result anticipates by almost two centuries subsequent discoveries of black holes.

Taking Einstein's equation (19) for a massive object and the space with a rotational symmetry, it can be shown that the spacetime metric could be written in the form

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{r_s}{r}\right)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (22)$$

that is,

$$ds^2 = -\left(\frac{r - r_s}{r}\right) dt^2 + \left(\frac{r}{r - r_s}\right) dr^2 + r^2 d\Omega^2. \quad (23)$$

This is the Schwarzschild solution 1916 (Schwarzschild, 1916a,b; Droste, 1917), and is the only possible spherically symmetric solution. Note that it depends only on the total mass of the body, in complete analogy to the Newtonian theory, because the r_s parameter is the same found by Laplace (21). At large distances from the origin, this metric behaves like

$$ds^2 \simeq ds_{Newt}^2 - \frac{r_s}{r}(dt^2 + dr^2), \quad (24)$$

that is, the Newtonian solution plus a perturbation of order r_s , thus at large distances every field appears centrally symmetric. At distances close to r_s , and $r > r_s$, the time slows down because of acceleration due to a strong gravitational field. The boundary at $r = r_s$ is called the event horizon, where the escape velocity is larger than the speed of light, as already seen in (21). If an object, or even radiation, crosses this boundary from the

external space, it can only eventually fall into the black hole. To provide some numbers, the Schwarzschild radius of our Sun is of the order of 3 km, while for the Earth it is less than 1 cm.

Observe that the Schwarzschild solution is singular only by the choice of coordinates that could be seen from the fact that the determinant of metric (23) is actually regular for $r = r_s$. The question was cleared only much later by Kruskal (Kruskal, 1960), where his coordinates are well defined at the event horizon $r = r_s$.

There are other solutions of Einstein's equation (19) that lead to black holes. A nonrotating charged black hole comes from the Reissner–Nordström metric (Reissner, 1916; Nordström, 1918).

Later, a rotating noncharged black hole solution came from the Kerr metric (Kerr, 1963).

Black hole properties

A star is essentially a ball of gas at equilibrium for most of its life. Inside its core nuclear reactions occur, where fusion of H nuclei into He create energy. The gravitation is in equilibrium with internal pressure due to heat as long as there is enough nuclear fuel. At the end of fuel, gravity prevails, the star crushes under its own weight, and a remnant is left.

If the remnant of the star is approximately larger than 3–4 M_\odot , the mass of the Sun, it is still too heavy and will eventually collapse to a black hole, with a radius smaller than the Schwarzschild one, $r < r_s$. There is no known mechanism that could prevent this process, not even in neutron stars, except maybe for quark stars, provided they exist at all.

From quantum mechanics (see for instance (Fabiano, 2022) for a discussion of the following subject), the probability amplitude of transition from an initial state $|in\rangle$ to a final state $|fi\rangle$ is given by the matrix element

$$\langle fi | e^{-itH} | in \rangle . \quad (25)$$

In thermodynamics, the probability of finding a state $|n\rangle$ of energy E_n at a given temperature T is given by Boltzmann:

$$\frac{\exp(-\beta E_n)}{Z} , \quad (26)$$

where $\beta = 1/T$ and Z is the partition function of the quantum mechanical system:

$$Z = \sum_{n=1}^{+\infty} \langle n | e^{-\beta E_n} | n \rangle = \sum_{n=1}^{+\infty} e^{-\beta E_n} = \text{Tr} e^{-\beta H} . \quad (27)$$

Setting the initial and final states to be the same, i.e. $|in\rangle = |fi\rangle = |n\rangle$, and replacing the time with the inverse of the temperature by means of a Wick rotation, that is, the exchange $it \leftrightarrow \beta$, one could identify the two formulae (25) and (27) to be the same expression. What concerns us most here is that with this procedure the time t becomes periodic and corresponds to an inverse of the system temperature, $1/T$.

Rewrite now the Schwarzschild metric (23) in the following manner. Choose ρ such that $d\rho^2 = r_s/(r - r_s)dr^2$, that is: $\rho^2 = 4r_s(r - r_s)$. Therefore, $\rho^2 d\rho^2 = 4r_s^2 dr^2 = (r - r_s)d\rho^2$.

Near the Schwarzschild radius one obtains:

$$ds^2 \approx -\frac{\rho^2}{4r_s^2} dt^2 + d\rho^2 + r_s^2 d\Omega^2 . \quad (28)$$

In Euclidean time, $t_E = it$ so $dt_E^2 = -dt^2$, therefore, the metric will gain all + signs:

$$ds^2 \approx \frac{\rho^2}{4r_s^2} dt_E^2 + d\rho^2 + r_s^2 d\Omega^2 . \quad (29)$$

Define ψ such that $t_E = 2r_s\psi$, obtaining finally the metric for $r \approx r_s$

$$ds^2 \approx \rho^2 d\psi^2 + d\rho^2 + r_s^2 d\Omega^2 . \quad (30)$$

By inspection of the result, the first two terms are radial coordinates on a plane, (ρ, ψ) , ψ having a 2π period, so from its definition follows that t_E has a period of $4\pi r_s$. With the above identification $t_E \leftrightarrow 1/T$, we have the following expression for the temperature of the black hole:

$$T = \frac{1}{4\pi r_s} = \frac{1}{8\pi GM} . \quad (31)$$

This is the Hawking temperature (Hawking, 1975, 1974), which is due to a quantum mechanical effect in the region near to the event horizon. Black holes spontaneously emit thermal radiation at this temperature. As seen from the derivation of T , it is due to vacuum fluctuations of quantum fields, which produce pairs of particles; one of them falls inside the black hole, while the other escapes to infinity.

The Hawking temperature increases when the mass of the black hole is smaller, and by radiating the black hole loses energy, i.e. it evaporates. It could be shown that the evaporation time for a black hole scales with the cube of its mass, i.e.

$$t_{evap} \propto M^3, \quad (32)$$

and while a black hole with the mass of our Sun will evaporate for a much longer time than the age of the Universe, a lighter one should still be observable by astronomers while exploding.

From its temperature, it is easy to find out the entropy of a black hole. From the expression of the energy, $E = c^2M$, that implies $dE = TdS = c^2dM$, thus obtaining the proportionality relations:

$$dS \propto \frac{1}{T}c^2dM \propto Gc^2d(M^2), \text{ therefore, } S \propto M^2. \quad (33)$$

The entropy of a black hole is proportional to the square of its mass, and from the expression of the Schwarzschild radius (21), $r_s \propto M$, therefore the entropy is proportional to r_s^2 , the area of the event horizon:

$$S \propto M \propto r_s^2 = A. \quad (34)$$

The entropy is an extensive quantity, proportional to the volume of the system considered. For a black hole, it is not so. The entropy is proportional to its area, not its volume.

This strange result was the inspiration of the so-called holographic principle, first proposed by Hooft and later formalized in string theory by Susskind (Susskind, 1995), for which gravity phenomena in a volume of space are actually projections of a two dimensional space, just like the holograms, which create three dimensional images from a two dimensional plate.

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Una breve introducción a los agujeros negros

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CAMPO: Física

TIPO DE ARTÍCULO: artículo de revisión

Resumen:

Introducción/objetivo: Partiendo de la relatividad general, se investiga la generación y los efectos de los agujeros negros.

Métodos: La ecuación de Einstein y su solución de Schwarzschild se emplean para estudiar los agujeros negros. Para obtener la radiación de Hawking se utiliza la mecánica cuántica.

Resultados: En realidad, los agujeros negros no son completamente negros: irradian energía durante su vida.

Conclusión: Los agujeros negros podrían evaporarse y este efecto es observable si su masa es lo suficientemente pequeña. Su entropía escala de manera diferente con respecto a su masa que la de otros objetos en termodinámica.

Palabras claves: relatividad general, agujeros negros, radiación de Hawking.

Краткое введение в черные дыры

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РУБРИКА ГРНТИ: 29.05.03 Математические методы
теоретической физики,
29.05.19 Специальная теория
относительности,
29.05.41 Гравитационное взаимодействие.
Общая теория относительности

ВИД СТАТЬИ: обзорная статья

Резюме:

Введение/цель: В данной статье объясняются образование и влияние черных дыр, начиная с общей теории относительности.

Методы: В изучении черных дыр используются уравнения Эйнштейна и метрика Шварцшильда в качестве решения этих уравнений. С помощью квантовой механики описано излучение Хокинга.

Результаты: Черные дыры на самом деле не являются полностью черными, они излучают энергию в течение своего существования.

Выводы: Черные дыры могут испаряться. Чем меньше масса черной дыры, тем заметнее ее излучение и исчезновение. Соотношение энтропии и массы черных дыр отличается от соотношения других термодинамических объектов.

Ключевые слова: общая теория относительности, черные дыры, излучение Хокинга.

Основна сазнања о црним рупама

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ОБЛАСТ: математика

КАТЕГОРИЈА ЧЛАНКА: прегледни рад

Сажетак:

Увод/циљ: У чланку су објашњени настанак и ефекти црних рупа почевши од опште теорије релативности.

Метод: За проучавање црних рупа искоришћене су Ајнштајнове једначине и Шварцшилдово решење тих једначи-

на. Помоћу квантне механике описано је Хокингово зрачење.

Резултати: Црне рупе нису у потпуности „црне”. Оне емитују енергију у виду зрачења током свог века трајања.

Закључак: Црне рупе могу испарити. Овај ефекат би могао бити видљив уколико је њихова маса довољно мала, а њихова ентропија се мења другачије од промене у ма-си у односу на друге термодинамичке објекте.

Кључне речи: општа теорија релативности, црне рупе, Хокингово зрачење.

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