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Variations in the Tensorial Trapezoid Type Inequalities for Convex Functions of Self-Adjoint Operators in Hilbert Spaces

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Abstract: In this paper, various tensorial inequalities of trapezoid type were obtained. Identity from classical analysis is utilized to obtain the tensorial version of the said identity which in turn allowed us to obtain tensorial inequalities in Hilbert space. The continuous functions of self-adjoint operators in Hilbert spaces have several tensorial norm inequalities discovered in this study. The convexity features of the mapping f lead to the variation in several inequalities of the trapezoid type.

Keywords: tensorial product; self-adjoint operators; convex functions

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1. Introduction

When Gibbs first conceptualized the idea of a tensor, he used the name "dyadic" instead of the official term "tensor". The mathematical version of the tensor concept is how we refer to it nowadays. Due to the extensive use of inequalities in mathematics, tensors can also benefit from their use. A major influence of inequalities is seen in mathematics and other scientific fields. There are many different kinds of inequalities, but those involving Jensen, Trapezoid, Hermite–Hadamard, and Minkowski are of particular importance. Interested readers can learn more about inequalities and their history in these books [1–3]. Regarding the generalizations of the aforementioned inequalities, numerous studies have been published; for additional information, check the following and the references therein [4–14].

A brief overview of the cited papers is given, so as to showcase the investigation of the said inequalities. In the paper written by Sarikaya et al. [4], the following Simpson type inequality is given, which is utilized to refine the inequality given in the corollary.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is a convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{1}$$
$$\leq \frac{5(b-a)}{72}[|f'(a)| + |f'(b)|].$$

Corollary 1. *Setting $f(a) = f(b) = f\left(\frac{a+b}{2}\right)$ in (1) refines the inequality given by Kirmaci [5]*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)| + |f'(b)|}{2}\right].$$

In the paper written by Hezenci et al. [6], the authors obtained fractional Simpson-type inequalities which generalized the inequalities obtained in the paper by Sarikaya et al. [4].

In the paper written by Özdemir and Ardic [7], various companion Ostrowski-type inequalities were given.

In the papers of Afzal et al. [8–10], the authors obtained several Hermite–Hadamard inequalities in various settings, such as the interval-valued inclusion being used in [8,9] and the center-radius inclusion being used in [10].

In the paper written by Butt et al. [11], a new type of convexity was introduced, namely the generalized exponential-type convexity which generalized previously known types of convexity, such as m–poly harmonically exp convex, exponential type convex function, and so on.

In the paper written by Chandola et al. [12], $k-p$ fractional Hermite–Hadamard-type inequalities were obtained. And as fractional $k-p$ operator generalizes the Riemann–Liouville fractional operator, the obtained inequalities of $k-p$ type generalize the ones obtained in the RL sense.

In the paper written by Chen and Katugampola [13], the authors utilized the Katugampola fractional operator to generalize the Hermite–Hadamard inequality as well as Ostrowski-type inequalities. And since Katugampola fractional operator generalizes the RL fractional operator, the inequalities obtained using the Katugampola operator given in [13] generalized all the inequalities obtained using the classical integral or using RL.

In the paper written by Stojiljković et al. [14], Hermite–Hadamard-type inequalities of tensorial type were obtained, which generalized the ones given by Dragomir [15].

Theorem 2. *Assume that ψ is a differentiable convex function on the interval I, and Υ and Ξ are self-adjoint operators with $Sp(\Upsilon), Sp(\Xi) \subset I$; then, for all $\tau \in (0,1]$, we have*

$$0 \leqslant (1-\xi)\psi\left(\frac{\Upsilon}{\tau}\right) \otimes 1 + \xi 1 \otimes \psi\left(\frac{\Xi}{\tau}\right) - \psi\left(\frac{1-\xi}{\tau}\Upsilon \otimes 1 + \xi 1 \otimes \Xi\right)$$

$$+\xi\left(1-\frac{1}{\tau}\right)\int_0^1 (1\otimes\Xi)\psi'\left(\frac{1-\xi u}{\tau}\Upsilon\otimes 1 + u\xi 1\otimes\Xi\right)du$$

$$\leqslant \left(\frac{\xi(1-\xi)}{\tau}\Upsilon\otimes 1 - \xi\cdot\left(\frac{1}{\tau}-\xi\right)1\otimes\Xi\right)\left(\psi'(\Upsilon)\otimes 1 - 1\otimes\psi'(\Xi)\right).$$

In particular,

$$0 \leqslant \frac{\psi(2\Upsilon)\otimes 1 + 1\otimes\psi(2\Xi)}{2} - \psi\left(\Upsilon\otimes 1 + \frac{1\otimes\Xi}{2}\right)$$

$$-\frac{1}{2}\int_0^1 (1\otimes\Xi)\psi'\left((2-t)\Upsilon\otimes 1 + \frac{t}{2}1\otimes\Xi\right)dt$$

$$\leqslant \left(\frac{\Upsilon\otimes 1}{2} - \frac{3}{4}1\otimes\Xi\right)\left(\psi'(\Upsilon)\otimes 1 - 1\otimes\psi'(\Xi)\right).$$

As the subject of our paper are tensorial Trapezoid-type inequalities, let us first provide a brief overview of the topic. The trapezoid inequality, as it is often called in the literature, is the following inequality:

Theorem 3. *Let $\mathfrak{f}$ be a mapping such that $\mathfrak{f} : [a,b] \to \mathbb{R}$ and $\mathfrak{f}$ is twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), that is $\|\mathfrak{f}''\|_{+\infty} = \sup_{x\in(a,b)} |\mathfrak{f}''(x)| < +\infty$. Then, the following inequality holds:*

$$\left|\int_a^b \mathfrak{f}(x)dx - \frac{\mathfrak{f}(a)+\mathfrak{f}(b)}{2}(b-a)\right| \leq \frac{1}{12}\|\mathfrak{f}''\|_{+\infty}(b-a)^3. \tag{2}$$

To enhance the presentation of this work, new developments in the theory of inequalities in Hilbert spaces will be demonstrated. Dragomir [16] gave the following inequality:

Theorem 4. *Let $(H;\langle .,.\rangle)$ be a Hilbert space and $T : H \mapsto H$ a normal linear operator on H. Then,*

$$\left(\|Tx\|^2\right) \geq \frac{1}{2}\left(\|Tx\|^2 + |\langle T^2x, x\rangle|\right) \geq |\langle Tx, x\rangle|^2,$$

for any $x \in H$, $\|x\| = 1$. The constant $\frac{1}{2}$ is the best possible.

The Hermite–Hadamard inequality in the self-adjoint operator sense, as provided by Dragomir [17], is another intriguing conclusion.

Theorem 5. *Let $\mathfrak{f} : I \to \mathbb{R}$ be an operator convex function on the interval I. Then, for any self-adjoint operators Υ and Ξ with spectra in I, we have the inequality*

$$\mathfrak{f}\left(\frac{\Upsilon+\Xi}{2}\right) \leq \mathfrak{f}\left(\frac{3\Upsilon+\Xi}{4}\right) + \mathfrak{f}\left(\frac{\Upsilon+3\Xi}{4}\right)$$

$$\leq \int_0^1 \mathfrak{f}((1-t)\Upsilon + t\Xi)dt$$

$$\leq \frac{1}{2}\left[\mathfrak{f}\left(\frac{\Upsilon+\Xi}{2}\right) + \frac{\mathfrak{f}(\Upsilon)+\mathfrak{f}(\Xi)}{2}\right] \leq \frac{\mathfrak{f}(\Upsilon)+\mathfrak{f}(\Xi)}{2}.$$

The first paper related to tensorial inequalities in Hilbert space was written by Dragomir [18]. In the paper, he proved the tensorial version of the Ostrowski-type inequality given by the following:

Theorem 6. *Assume that $\mathfrak{f}$ is continuously differentiable on I with $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I} |\mathfrak{f}'(t)| < +\infty$ and A, B are self-adjoint operators with $Sp(\Upsilon), Sp(\Xi) \subset I$. Then, the following inequality holds:*

$$\left\|\mathfrak{f}((1-\lambda)\Upsilon\otimes 1 + \lambda 1\otimes\Xi) - \int_0^1 \mathfrak{f}((1-u)\Upsilon\otimes 1 + u1\otimes\Xi)du\right\| \tag{3}$$

$$\leq \|\mathfrak{f}'\|_{I,+\infty}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes\Xi - \Upsilon\otimes 1\|$$

for $\lambda \in [0,1]$.

Recently, various inequalities in the same tensorial surrounding have been obtained. The following result of Simpson type was obtained by Stojiljković [19].

Theorem 7. *Assume that $\mathfrak{f}$ is continuously differentiable on I and $|\mathfrak{f}''|$ is convex and A, B are self-adjoint operators with $Sp(\Upsilon), Sp(\Xi) \subset I$. Then, the following inequality holds:*

$$\Bigg\|\frac{1}{6}\left(\mathfrak{f}(\Upsilon)\otimes 1 + 4\mathfrak{f}\left(\frac{\Upsilon\otimes 1 + 1\otimes\Xi}{2}\right) + 1\otimes\mathfrak{f}(\Xi)\right)$$

$$-\frac{1}{2}\alpha\left(\int_0^1 \mathfrak{f}\left(\left(\frac{1-k}{2}\right)\Upsilon\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes\Xi\right)k^{\alpha-1}dk\right.$$

$$\left.+\int_0^1 \mathfrak{f}\left(\left(1-\frac{k}{2}\right)\Upsilon\otimes 1 + \frac{k}{2}1\otimes\Xi\right)(1-k)^{\alpha-1}dk\right)\Bigg\|$$

$$\leq \|1\otimes\Xi - \Upsilon\otimes 1\|^2 \frac{(\|\mathfrak{f}''(\Upsilon)\| + \|\mathfrak{f}''(\Xi)\|)(3\alpha^2 + 8\alpha + 7)}{(\alpha+2)(24\alpha+24)}$$

for $\alpha \geq 0$.

The following inequality has been recently obtained by the same author [20].

Theorem 8. *Formulation is the same as the one given by Dragomir in his Ostrowski-type Theorem given above* (3)*. Then, the following inequality holds:*

$$\left\| \int_0^1 \mathfrak{f}((1-\lambda)\Upsilon \otimes 1 + \lambda 1 \otimes \Xi) d\lambda - \mathfrak{f}\left(\frac{\Upsilon \otimes 1 + 1 \otimes \Xi}{2} \right) \right\|$$

$$\leqslant \|1 \otimes \Xi - \Upsilon \otimes 1\|^2 \frac{\|\mathfrak{f}''\|_{I,+\infty}}{24}.$$

Recently, the following inequality of Ostrowski type was obtained by Stojiljković et al. [21] which generalized the recently obtained results by Dragomir [18].

Theorem 9. *The formulation is the same as the one given by Dragomir in his Ostrowski-type theorem given above* (3) *with an exception that $\alpha > 0$; then,*

$$\Bigg\| \left(\lambda^\alpha + (1-\lambda)^\alpha \right) f((1-\lambda)\Upsilon \otimes 1 + \lambda 1 \otimes \Xi)$$

$$-\alpha \Bigg((1-\lambda)^\alpha \int_0^1 f((1-\lambda)(1-u)\Upsilon \otimes 1 + (u + (1-u)\lambda) 1 \otimes \Xi)(1-u)^{\alpha-1} du$$

$$+\lambda^\alpha \int_0^1 u^{\alpha-1} f(((1-u) + u(1-\lambda))\Upsilon \otimes 1 + u\lambda 1 \otimes \Xi) du \Bigg) \Bigg\|$$

$$\leqslant \|1 \otimes \Xi - \Upsilon \otimes 1\| \left(\frac{\lambda^{\alpha+1}}{\alpha+1} + \frac{(1-\lambda)^{\alpha+1}}{\alpha+1} \right) \|f'\|_{I,+\infty}.$$

2. Preliminaries

We require these preliminary steps and the subsequent introduction in order to derive equivalent inequalities of the tensorial type.

Let $I_1, \ldots, I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times \ldots \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $\Upsilon = (\Upsilon_1, \ldots, \Upsilon_k)$ be a k-tuple of bounded self-adjoint operators on Hilbert spaces $H_1, \ldots, H_k$ such that the spectrum of Υ_i is contained in I_i for $i = 1, \ldots, k$. We say that such a k-tuple is in the domain of f. If

$$\Upsilon_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of Υ_i for $i = 1, \ldots, k$ by following this, we define

$$f(\Upsilon_1, \ldots, \Upsilon_k) := \int_{I_1} \cdots \int_{I_k} f(\lambda_1, \ldots, \lambda_k) dE_1(\lambda_1) \otimes \ldots \otimes dE_k(\lambda_k)$$

as a bounded self-adjoint operator on the tensorial product $H_1 \otimes \ldots H_k$.

Detailed introduction and construction can be found in a paper written by Dragomir [18], for more information also see [22,23].

Remember the following properties of the tensorial product:

$$(\Upsilon \mathfrak{C}) \otimes (\Xi \mathfrak{D}) = (\Upsilon \otimes \Xi)(\mathfrak{C} \otimes \mathfrak{D})$$

which holds for any $\Upsilon, \Xi, \mathfrak{C}, \mathfrak{D} \in B(H)$.

From the property, we can easily deduce the following consequences:

$$\Upsilon^n \otimes \Xi^n = (\Upsilon \otimes \Xi)^n, n \geqslant 0,$$

$$(\Upsilon \otimes 1)(1 \otimes \Xi) = (1 \otimes \Xi)(\Upsilon \otimes 1) = \Upsilon \otimes \Xi,$$

which can be extended; for two natural numbers r, s, we have

$$(\Upsilon \otimes 1)^s (1 \otimes \Xi)^r = (1 \otimes \Xi)^r (\Upsilon \otimes 1)^s = \Upsilon^s \otimes \Xi^r.$$

For more information about the tensorial products and the usage of them in this paper, see the paper written by Dragomir [18] and the book given [24].

We need the following Lemma, which is in a paper by Dragomir [25].

Lemma 1. *Assume Υ and Ξ are self-adjoint operators with $Sp(\Upsilon) \subset I, Sp(\Xi) \subset J$ and with spectral resolutions. Let $f; h$ be continuous on I, g, k continuous on J, and ϕ and ψ continuous on an interval K that contains the sum of the intervals $f(I) + g(J); h(I) + k(J)$; then,*

$$\phi(f(\Upsilon) \otimes 1 + 1 \otimes g(\Xi))\psi(h(\Upsilon) \otimes 1 + 1 \otimes k(\Xi))$$
$$= \int_I \int_J \phi(f(t) + g(s))\psi(h(t) + k(s)) dE_t \otimes dF_s.$$

We require the following definitions of the fractional integral:

Definition 1. *Let $f \in C([a, b])$. Then, the left and right sided Riemann–Liouville (RL) fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as*

$${}_{RL}J^{\alpha}_{a+} f(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z - u)^{\alpha - 1} f(u) du, z > a \tag{4}$$

$${}_{RL}J^{\alpha}_{b-} f(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (u - z)^{\alpha - 1} f(u) du, z < b, \tag{5}$$

where $\Gamma(.)$ denotes the Gamma function given by $\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1}, \Re(s) > 0$.

Lemma 2. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f(b) + f(a)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [{}_{RL}J^{\alpha}_{a+} f(b) +_{RL} J^{\alpha}_{b-} f(a)] \tag{6}$$
$$= \frac{(b - a)}{2} \int_0^1 [(1 - t)^{\alpha} - t^{\alpha}] f'(ta + (1 - t)b) dt.$$

Sarikaya et al. [26] established new Ostrowski-type inequalities for Riemann–Liouville fractional integrals (see Theorem 3), which generalize the classical Ostrowski-type inequality by utilizing the previously established integral identity (6) via convex mappings.

Novel aspects in this work can be seen in the development of the inequalities of the Trapezoid type for the differentiable functions in the Hilbert space of tensorial type. This field is relatively new; therefore, obtaining new bounds for various convex combinations of the functions is instrumental to the development of it. The rest of this paper is structured as follows: the main results is the section in which the results concerning the novelty of the work will be given. In the following section, some examples and consequences will feature examples of the obtained results by using the known fact about the exponential operator and its integral; therefore, by utilizing it and choosing f to be a specific convex function, we obtain numerous examples and bounds of the Trapezoid type in the tensorial sense. In the Conclusion Section, we conclude on what has been found. We provide a basic result in the following theorem, which we will utilize in our article to obtain inequalities.

3. Main Results

In this section, we first demonstrate a tensorial Lemma which we will utilize throughout the paper.

Lemma 3. *Assume that f is continuously differentiable on I, A and Ξ are self-adjoint operators with $Sp(\Upsilon), Sp(\Xi) \subset I$. Then, the following equality holds:*

$$
\begin{aligned}
&(f(\Upsilon)\otimes 1 + 1\otimes f(\Xi)) \qquad (7)\\
-\alpha&\left[\int_0^1 (1-\lambda)^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda + \int_0^1 \lambda^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda\right]\\
&= (1\otimes\Xi - \Upsilon\otimes 1)\int_0^1 \left[(1-\zeta)^\alpha - \zeta^\alpha\right] f'(\zeta\Upsilon\otimes 1 + (1-\zeta)1\otimes\Xi)d\zeta.
\end{aligned}
$$

Proof. We will start the proof with Lemma (6). Introducing a substitution on the left-hand side, namely $x = \lambda b + (1-\lambda)a$, simplifying and then assuming that Υ and Ξ have spectral resolutions

$$\Upsilon = \int_I t dE(t) \ and \ \Xi = \int_I s dF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we obtain

$$
\begin{aligned}
&\int_I\int_I (f(t) + f(s)) dE_t\otimes dF_s\\
-\alpha&\left[\int_I\int_I\left(\int_0^1 (1-\lambda)^{\alpha-1} f(\lambda s + (1-\lambda)t)d\lambda + \int_0^1 \lambda^{\alpha-1} f(\lambda s + (1-\lambda)t)d\lambda\right)dE_t\otimes dF_s\right]\\
&= \int_I\int_I\left((s-t)\int_0^1\left[(1-\zeta)^\alpha - \zeta^\alpha\right] f'(\zeta t + (1-\zeta)s)d\zeta\right)dE_t\otimes dF_s.
\end{aligned}
$$

By utilizing the Fubinis Theorem and Lemma 1 for appropriate choices of the functions involved, we have, successively,

$$
\begin{aligned}
\int_I\int_I\int_0^1 f((1-\lambda)t + \lambda s)d\lambda dE_t\otimes dF_s &= \int_0^1\int_I\int_I f((1-\lambda)t + \lambda s)dE_t\otimes dF_s d\lambda\\
&= \int_0^1 f((1-\lambda)\Upsilon\otimes 1 + \lambda 1\otimes\Xi)d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
&\int_I\int_I (s-t)\int_0^1 [(1-\zeta)^\alpha - \zeta^\alpha] f'(\zeta t + (1-\zeta)s)d\zeta dE_t\otimes dF_s\\
&= \int_0^1 [(1-\zeta)^\alpha - \zeta^\alpha]\int_I\int_I (s-t) f'(\zeta t + (1-\zeta)s)dE_t\otimes dF_s d\zeta\\
&= \int_0^1 [(1-\zeta)^\alpha - \zeta^\alpha](1\otimes\Xi - \Upsilon\otimes 1) f'(\lambda\Upsilon\otimes 1 + (1-\lambda)1\otimes\Xi)d\zeta.
\end{aligned}
$$

□

Theorem 10. *Formulation is the same as the one given by Dragomir in his Ostrowski-type Theorem given above (3). Then, the following inequality holds:*

$$
\begin{aligned}
&\Bigg\|(f(\Upsilon)\otimes 1 + 1\otimes f(\Xi)) \qquad (8)\\
-\alpha&\left[\int_0^1 (1-\lambda)^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda + \int_0^1 \lambda^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda\right]\Bigg\|\\
&\le \|1\otimes\Xi - \Upsilon\otimes 1\|\frac{1}{1+\alpha}\left(2 - 2^{1-\alpha}\right)\|f'\|_{I,+\infty}.
\end{aligned}
$$

Proof. If we take the operator norm of the previously obtained Lemma (7) and use the triangle inequality, we obtain

$$\Bigg\|(f(\Upsilon)\otimes 1 + 1\otimes f(\Xi))$$

$$-\alpha\left[\int_0^1 (1-\lambda)^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda + \int_0^1 \lambda^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda\right]\Bigg\|$$

$$\leqslant \|1\otimes\Xi - \Upsilon\otimes 1\| \int_0^1 \Bigg\|\Big[(1-\zeta)^\alpha - \zeta^\alpha\Big]\Bigg\| \big\|f'(\zeta\Upsilon\otimes 1 + (1-\zeta)1\otimes\Xi)\big\| d\zeta.$$

Realize here that by Lemma 1,

$$\Big|f'(\zeta\Upsilon\otimes 1 + (1-\zeta)1\otimes\Xi)\Big| = \int_I\int_I \Big|f'(\zeta t + (1-\zeta)s)\Big| dE_t\otimes dF_s.$$

Since

$$\Big|f'(\zeta t + (1-\zeta)s)\Big| \leqslant \|f'\|_{I,+\infty},$$

for all $t,s\in I$. If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we obtain

$$\Big|f'(\zeta\Upsilon\otimes 1 + (1-\zeta)1\otimes\Xi)\Big| = \int_I\int_I \Big|f'(\zeta t + (1-\zeta)s)\Big| dE_t\otimes dF_s$$

$$\leqslant \|f'\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s = \|f'\|_{I,+\infty}.$$

From which we obtain the following:

$$\int_0^1 \|(1-\zeta)^\alpha - \zeta^\alpha\| \big\|f'(\zeta\Upsilon\otimes 1 + (1-\zeta)1\otimes\Xi)\big\| d\lambda$$

$$\leqslant \|f'\|_{I,+\infty}\int_0^1 \|(1-\zeta)^\alpha - \zeta^\alpha\| d\zeta = \frac{1}{1+\alpha}(2-2^{1-\alpha})\|f'\|_{I,+\infty}.$$

□

Theorem 11. *Assume that f is continuously differentiable on I and $|f'|$ is convex and A,B are self-adjoint operators with $Sp(\Upsilon), Sp(\Xi)\subset I$. Then, the following inequality holds:*

$$\Bigg\|(f(\Upsilon)\otimes 1 + 1\otimes f(\Xi)) \tag{9}$$

$$-\alpha\left[\int_0^1 (1-\lambda)^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda + \int_0^1 \lambda^{\alpha-1} f(\lambda 1\otimes\Xi + (1-\lambda)\Upsilon\otimes 1)d\lambda\right]\Bigg\|$$

$$\leqslant \|1\otimes\Xi - \Upsilon\otimes 1\| \frac{2^{-\alpha}(2^\alpha - 1)(\|f'(\Upsilon)\| + \|f'(\Xi)\|)}{(\alpha+1)}.$$

Proof. Since $|f'|$ is convex on I, then we obtain

$$\Big|f'(\zeta t + (1-\zeta)s)\Big| \leqslant \zeta|f'(t)| + (1-\zeta)|f'(s)|$$

for all $\zeta\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we obtain

$$\Big|f'(\zeta\Upsilon\otimes 1 + (1-\zeta)1\otimes\Xi)\Big| = \int_I\int_I \Big|f'(\zeta t + (1-\zeta)s)\Big| dE_t\otimes dF_s$$

$$\leqslant \int_I \int_I \left[\zeta|f'(t)| + (1-\zeta)|f'(s)|\right] dE_t \otimes dF_s$$
$$= \zeta|f'(\Upsilon)| \otimes 1 + (1-\zeta)1 \otimes |f'(\Xi)|$$

for all $\zeta \in [0,1]$.

If we take the norm in the inequality, we obtain the following

$$\left\|f'(\zeta\Upsilon \otimes 1 + (1-\zeta)1 \otimes \Xi)\right\| \leqslant \left\|\zeta|f'(\Upsilon)| \otimes 1 + (1-\zeta)1 \otimes |f'(\Xi)|\right\|$$
$$\leqslant \zeta\left\||f'(\Upsilon)| \otimes 1\right\| + (1-\zeta)\left\|1 \otimes |f'(\Xi)|\right\|$$
$$= \zeta\left\|f'(\Upsilon)\right\| + (1-\zeta)\left\|f'(\Xi)\right\|.$$

Therefore, we obtain

$$\int_0^1 \|(1-\zeta)^\alpha - \zeta^\alpha\| \left\|f'(\zeta\Upsilon \otimes 1 + (1-\zeta)1 \otimes \Xi)\right\| d\zeta$$
$$\leqslant \int_0^1 \|(1-\zeta)^\alpha - \zeta^\alpha\| (\zeta\left\|f'(\Upsilon)\right\| + (1-\zeta)\left\|f'(\Xi)\right\|) d\zeta.$$
$$= \frac{2^{-\alpha}(2^\alpha - 1)(\|f'(\Upsilon)\| + \|f'(\Xi)\|)}{\alpha+1}.$$

□

As we remember, the function $f : I \to \mathbb{R}$ is quasi-convex, if

$$f((1-\zeta)t + \zeta s) \leqslant \max(f(t), f(s)) = \frac{1}{2}(f(t) + f(s) + |f(s) - f(t)|)$$

holds for all $t, s \in I$ and $\zeta \in [0,1]$.

Theorem 12. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, Υ and Ξ are self-adjoint operators with $Sp(\Upsilon), Sp(\Xi) \subset I$. Then, the following inequality holds:*

$$\left\|\left|(f(\Upsilon) \otimes 1 + 1 \otimes f(\Xi))\right.\right. \tag{10}$$
$$\left.\left.-\alpha\left[\int_0^1 (1-\lambda)^{\alpha-1} f(\lambda 1 \otimes \Xi + (1-\lambda)\Upsilon \otimes 1) d\lambda + \int_0^1 \lambda^{\alpha-1} f(\lambda 1 \otimes \Xi + (1-\lambda)\Upsilon \otimes 1) d\lambda\right]\right\|\right.$$
$$\leqslant \frac{\|1 \otimes \Xi - \Upsilon \otimes 1\|}{2(1+\alpha)}(2 - 2^{1-\alpha})(\left\||f'(\Upsilon)| \otimes 1 + 1 \otimes |f'(\Xi)|\right\| + \left\||f'(\Upsilon)| \otimes 1 - 1 \otimes |f'(\Xi)|\right\|).$$

Proof. Since $|f'|$ is quasi-convex on I, then we obtain

$$\left|f'(\zeta t + (1-\zeta)s)\right| \leqslant \frac{1}{2}(|f'(t)| + |f'(s)| + ||f'(t)| - |f'(s)||)$$

for all $\zeta \in [0,1]$ and $t, s \in I$. If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we obtain

$$\left|f'(\zeta\Upsilon \otimes 1 + (1-\zeta)1 \otimes \Xi)\right|$$
$$= \int_I \int_I |f'(\zeta t + (1-\zeta)s)| dE_t \otimes dF_s$$
$$\leqslant \frac{1}{2} \int_I \int_I (|f'(t)| + |f'(s)| + ||f'(t)| - |f'(s)||) dE_t \otimes dF_s$$
$$= \frac{1}{2}(|f'(\Upsilon)| \otimes 1 + 1 \otimes |f'(\Xi)| + ||f'(\Upsilon)| \otimes 1 - 1 \otimes |f'(\Xi)||)$$

for all $\zeta \in [0,1]$.

If we take the norm, then we obtain

$$\begin{aligned}
&\left\| f'(\zeta \Upsilon \otimes 1 + (1-\zeta) 1 \otimes \Xi) \right\| \\
&\leqslant \left\| \frac{1}{2}(|f'(\Upsilon)| \otimes 1 + 1 \otimes |f'(\Xi)| + ||f'(\Upsilon)| \otimes 1 - 1 \otimes |f'(\Xi)||) \right\| \\
&\leqslant \frac{1}{2}(\||f'(\Upsilon)| \otimes 1 + 1 \otimes |f'(\Xi)|\| + \||f'(\Upsilon)| \otimes 1 - 1 \otimes |f'(\Xi)|\|)
\end{aligned}$$

Which, when applied in our case, obtains

$$\begin{aligned}
&\int_0^1 \|(1-\zeta)^\alpha - \zeta^\alpha\| \left\| f'(\zeta \Upsilon \otimes 1 + (1-\zeta) 1 \otimes \Xi) \right\| d\zeta \\
&\leqslant \int_0^1 \|(1-\zeta)^\alpha - \zeta^\alpha\| \left(\frac{1}{2}(\||f'(\Upsilon)| \otimes 1 + 1 \otimes |f'(\Xi)|\| + \||f'(\Upsilon)| \otimes 1 - 1 \otimes |f'(\Xi)|\|) \right) d\zeta.
\end{aligned}$$

This, when simplified, yields the intended inequality. $\square$

4. Some Examples and Consequences

Corollary 2. *If* Υ, Ξ *are self-adjoint operators with* $Sp(\Upsilon), Sp(\Xi) \subset [m, M]$*; then, by* (8)*, we obtain*

$$\begin{aligned}
&\Bigg\| (\exp(\Upsilon) \otimes 1 + 1 \otimes \exp(\Xi)) \\
&-\alpha \left[\int_0^1 (1-\lambda)^{\alpha-1} \exp(\lambda 1 \otimes \Xi + (1-\lambda) \Upsilon \otimes 1) d\lambda + \int_0^1 \lambda^{\alpha-1} \exp(\lambda 1 \otimes \Xi + (1-\lambda) \Upsilon \otimes 1) d\lambda \right] \Bigg\| \\
&\leq \|1 \otimes \Xi - \Upsilon \otimes 1\| \frac{1}{1+\alpha} \left(2 - 2^{1-\alpha}\right) \exp(M).
\end{aligned} \tag{11}$$

Corollary 3. *Since for* $f(t) = \exp(t), t \in \mathbb{R}$, $|f'|$ *is convex; then, by* (9)

$$\begin{aligned}
&\Bigg\| (\exp(\Upsilon) \otimes 1 + 1 \otimes \exp(\Xi)) \\
&-\alpha \left[\int_0^1 (1-\lambda)^{\alpha-1} \exp(\lambda 1 \otimes \Xi + (1-\lambda) \Upsilon \otimes 1) d\lambda + \int_0^1 \lambda^{\alpha-1} \exp(\lambda 1 \otimes \Xi + (1-\lambda) \Upsilon \otimes 1) d\lambda \right] \Bigg\| \\
&\leqslant \|1 \otimes \Xi - \Upsilon \otimes 1\| \frac{2^{-\alpha}(2^\alpha - 1)(\|\exp(\Upsilon)\| + \|\exp(\Xi)\|)}{(\alpha+1)}.
\end{aligned} \tag{12}$$

Setting $\alpha = \frac{1}{2}$*, we obtain*

$$\begin{aligned}
&\Bigg\| (\exp(\Upsilon) \otimes 1 + 1 \otimes \exp(\Xi)) \\
&-\frac{1}{2} \left[\int_0^1 (1-\lambda)^{-\frac{1}{2}} \exp(\lambda 1 \otimes \Xi + (1-\lambda) \Upsilon \otimes 1) d\lambda + \int_0^1 \lambda^{-\frac{1}{2}} \exp(\lambda 1 \otimes \Xi + (1-\lambda) \Upsilon \otimes 1) d\lambda \right] \Bigg\| \\
&\leqslant \|1 \otimes \Xi - \Upsilon \otimes 1\| \frac{(2-\sqrt{2})(\|\exp(\Upsilon)\| + \|\exp(\Xi)\|)}{3}.
\end{aligned} \tag{13}$$

5. Conclusions

Because they offer a clear mathematical foundation for expressing and resolving physical problems in a variety of disciplines, including mechanics, electromagnetic, quantum mechanics, and many more, tensors have gained importance in a number of subjects, including physics. Inequalities are therefore essential in numerical aspects. Reflected in this work is the tensorial Sarikaya's Lemma, which allowed us to obtain Trapezoid-type inequalities in Hilbert space as a consequence. New Trapezoid-type inequalities were given, and some examples of specific convex functions and their inequalities using our results were given. Interplay between the classical theory of inequalities and tensorial inequalities in Hilbert space has been shown, as we used the Lemma given by Sarikaya in classical analysis which we transformed into a tensorial version of it which we then used to obtain new inequalities in the tensorial sense. Further directions in this area can be seen in obtaining various other classical identities for inequalities which can be used to turn them into tensorial ones which then would be used to obtain further inequalities and sharpen the existing ones. The fact that the inequalities discovered in this work can be refined or extended through alternative approaches can be interpreted as a sign of future research plans. An interesting perspective is provided by combining the techniques used in this study with other methods for Hilbert space inequalities. We will focus on the Mond-Pecaric inequality technique as one direction.

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