Decomposition of multi-particle azimuthal correlations in Q-cumulant analysis*

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Abstract: The method of Q-cumulants is a powerful tool for studying the fine details of azimuthal anisotropies in high energy nuclear collisions. This paper presents a new method, based on mathematical induction, to evaluate the analytical form of high-order Q-cumulants. The capability of this method is demonstrated via a toy model that uses the elliptic power distribution to simulate the anisotropic emission of particles, quantified in terms of Fourier flow harmonics v_n. The method can help in studying the large amount of event statistics that can be collected in the future and allow measurements of the very high central moments of the v_2 distribution. This can, in turn, facilitate progress in understanding the initial geometry, the input to the hydrodynamic calculations of medium expansion in high energy nuclear collisions, and the constraints on it.

Keywords: quark gluon plasma, azimuthal anisotropies, Q-cumulants, Pfaff transformation

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I. INTRODUCTION

In ultra-relativistic collisions of nuclei at both the Relativistic Heavy Ion Collider (RHIC) [1−4] and Large Hadron Collider (LHC) [5−8], a hot and dense system of strongly interacting quarks and gluons, known as “quark-gluon plasma” (QGP), is created. One of the key observables used to study the properties of QGP is the anisotropic collective flow, quantified by Fourier harmonics v_n. Many methods have been developed to measure these harmonics [9−11]. One such method, the method of cumulants based on multi-particle correlations [12, 13], enables the suppression of short-range correlations arising from jets and resonance decays and then reveals a genuine collectivity arising from the expansion of QGP. The Q-cumulant method [14] is an improved version of the initial cumulant method. Recently, Q-cumulants were employed to examine hydrodynamics predictions in Ref. [15] based on the ratio of the differences between adjacent cumulants [16]. This analysis, performed using CMS data, indicated the necessity of introducing moments higher than skewness to describe the finer details of the elliptic flow v_2 distribution. To do so, higher-order cumulants must be calculated. In this paper, we present a way of determining the expressions required to calculate higher-order cumulants.

Sec. II of the paper gives the basic quantities used in the Q-cumulant method. Sec. III presents the foundation of the method within a finite-dimensional vector space. Sec. IV shows the results along with applications of the method to simple cases. Using toy models to demonstrate the capabilities of the method in Sec. V, and a summary is given in Sec. VI. Tables that summarize the coefficients needed to express the cumulants up to the 14-th order can be found in the appendix.

II. BASICS OF Q-CUMULANTS

The general formalism of cumulants, for the purpose of flow measurements, was first used in Refs. [12−14], in which cumulants were expressed in terms of the moments of the magnitude of the corresponding flow vector. Such a cumulant method is systematically biased owing to the trivial effects of same particle autocorrelations. To

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avoid the computationally expensive "nested loops" used to remove autocorrelations in the measurements of multi-particle azimuthal correlations, Refs. [12−14] proposed an alternative method based on the use of generating functions. An improved cumulant method, Q-cumulants [14], allows, in principle, for fast and exact calculation of all multi-particle cumulants. However, in practice, the determination of the analytical expressions for multi-particle cumulants, which involve the azimuthal correlations of more than eight particles, is challenging.

Q-cumulants are built upon analytical expressions for multi-particle azimuthal correlations in terms of flow vectors Q, with

$$Q_n = \sum_{j=1}^{M} \mathrm{e}^{in\varphi_j}, \tag{1}$$

evaluated at different orders of Fourier harmonics, n. Here, M is the multiplicity or number of particles in an event, and φ_j denotes the azimuthal angle of the j-th particle measured in a fixed coordinate system in the laboratory. The method involves the decomposition of the $2m$-th power of the magnitude of the flow vector, $|Q_n|^{2m}$,

$$|Q_n|^{2m} = \sum_{j_1,\ldots,j_{2m}=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}+\ldots+\varphi_{j_m}-\varphi_{j_{m+1}}-\ldots-\varphi_{j_{2m}}\right)}, \tag{2}$$

into (off diagonal) terms with $2m$, $2m$-1, … different indices up to the term where all $2m$ indices are equal (diagonal term). The first term, with $2m$ different indices, of decomposition is proportional to the $2m$-particle azimuthal correlations $\langle 2m\rangle$,

$$\sum_{j_1\neq\ldots\neq j_{2m}=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}+\ldots+\varphi_{j_m}-\varphi_{j_{m+1}}-\ldots-\varphi_{j_{2m}}\right)} \equiv P_{M,2m}\cdot\langle 2m\rangle, \tag{3}$$

where $P_{M,2m}$ is the number of distinct $2m$-particle combinations one can form for an event with multiplicity M:

$$P_{M,2m} = \frac{M!}{(M-2m)!}. \tag{4}$$

In the case of full decomposition, $|Q_n|^{2m}$ is expressed in readily calculable terms of powers of the flow vector given with Eq. (1) along with the anticipated $2m$-particle azimuthal correlations introduced in Eq. (3). A thorough derivation with detailed examples is presented in Ref. [17]. The analytical decomposition of $|Q_n|^{2m}$ for m > 4, however, becomes tedious.

In this paper, we present a numerical method for the decomposition of $|Q_n|^{2m}$ that enables us, with the use of modern computers, to easily obtain analytical expressions for multi-particle azimuthal correlations of higher orders.

III. FOUNDATION OF THE METHOD

The full decomposition of $|Q_n|^{2m}$ is divided into appropriate sets (or multisets in the case of repeated elements) of azimuthal angles that lead to the partial Bell polynomials or their generalization (in the case of multisets). In the derivation, some parts of the Bell polynomials must be expressed through lower-order multi-particle azimuthal correlations. This change of side in the expression causes a sign change of the corresponding term, resulting in the coefficients of the partial Bell polynomials in the final form of the $2m$-particle azimuthal correlations being positive or negative integers. For example, in the case of m = 2, the four-particle azimuthal correlations might be presented as a linear combination of the corresponding Q-vector terms,

$$P_{M,4}\langle 4\rangle = +1|Q_n|^4 - 2\mathrm{Re}\left[Q_{2n}Q_n^*Q_n^*\right] + 1|Q_{2n}|^2 - 4(M-2)|Q_n|^2 + 2M(M-3), \tag{5}$$

with the corresponding integer coefficients (+1, –2, +1, −4, +2) in front of each of the terms.

The same holds true for all $2m$-particle correlations but with different sets of integer coefficients. This fact inspires us to calculate all these integer coefficients by solving an appropriate system of algebraic equations.

The $2m$-particle correlations $\langle 2m\rangle$ may be considered members of a finite-dimensional vector space V_d, where $d = \frac{1}{2}\sum_{l=0}^{m} p(l)[p(l)+1]$ is a dimension of the vector space, and $p(l) = \{1, 1, 2, 5, 7,\ldots\}$ is a partition function of a non-negative integer $l = \{0, 1, 2, 3, 4,\ldots\}$. In V_d, we may define a basis $B_{\langle 2m\rangle} = (\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\ldots,\mathbf{e}_d)$, which enables us to present $P_{M,2m}\langle 2m\rangle$ as a linear combination of the basis vectors $\mathbf{e}_i$:

$$P_{M,2m}\langle 2m\rangle = x_1 f_1^{(m,l)}\mathbf{e}_1 + x_2 f_2^{(m,l)}\mathbf{e}_2 + \ldots + x_d f_d^{(m,l)}\mathbf{e}_d, \tag{6}$$

where x_1, x_2, … are unknown integer numbers that must be obtained, and $f_i^{(m,l)}$ are known integer functions of the multiplicity M. The left side of Eq. (6) can be calculated directly from Eq. (3) only in case of low multiplicity because this is computationally manageable. The d unknown integer numbers, $x_1, x_2, \ldots, x_d$ on the right hand side of Eq. (6) may be calculated numerically by solving the system of d linear algebraic equations presented in the next section.

IV. DETERMINATION OF THE BASIS $B_{\langle 2m\rangle} = (\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\ldots,\mathbf{e}_d)$

The full analytical determination of the basis $B_{\langle 2m\rangle}$ is presented in Ref. [17] for $m = 1,\ldots,4$. However, in this paper, we show the straightforward determination of the basis using the method of mathematical induction, which enables easy calculation of the multi-particle azimuthal correlations of higher orders. First, it should be noted from simple inspection of the published results [14, 17] that each basis $B_{\langle 2m\rangle}$ contains the complete basis of the lower number (2m-2)-particle azimuthal correlations $B_{\langle 2m-2\rangle} \subset B_{\langle 2m\rangle}$. For example, the basis of the 4-particle azimuthal correlations contains the complete basis of the 2-particle azimuthal correlations $B_{\langle 2\rangle} = \left(1, |Q_n|^2\right)$ and some additional basis vectors,

$$B_{\langle 4\rangle} = \left(B_{\langle 2\rangle}, |Q_n|^4, \mathrm{Re}(Q_{2n}Q_n^*Q_n^*), |Q_{2n}|^2\right). \quad (7)$$

Moreover, the basis $B_{\langle 2\rangle} = \left(B_{\langle 0\rangle}, |Q_n|^2\right)$ might be considered an extension of the basis $B_{\langle 0\rangle}$, with an additional basis vector $|Q_n|^2$. All additional vectors in the basis $B_{\langle 2l\rangle}$ form a subset l, whose dimension is $d(l) = \frac{1}{2}p(l)\left[p(l)+1\right]$. Thus, the problem of determining the basis $B_{\langle 2m\rangle}$ reduces to finding the subset $l = m$.

We find that the subset $l = m$ contains different compositions (products) of the flow vectors $Q_{sub1}Q_{sub2}\cdots$, each with a subscript ($sub1$, $sub2$,...) that corresponds to the well-known "partition of the positive integer" m. For example, in the case of the 8-particle ($2m = 8$) azimuthal correlations, the subset $l = 4$ contains the following compositions of the flow vectors:

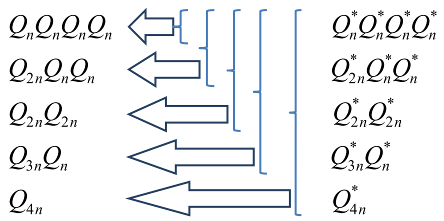
$$\begin{array}{l} Q_{1n}Q_{1n}Q_{1n}Q_{1n} \\ Q_{2n}Q_{1n}Q_{1n} \\ Q_{2n}Q_{2n} \\ Q_{3n}Q_{1n} \\ Q_{4n} \end{array} \quad \Leftarrow \quad \begin{cases} 1 \quad 1 \quad 1 \quad 1 \\ 2 \quad 1 \quad 1 \\ 2 \quad 2 \\ 3 \quad 1 \\ 4 \end{cases}, \quad (8)$$

where each composition contains subscripts that correspond to the partition of the integer 4.

Here, we use the convenient symbolic notation introduced in Refs. [14, 17],

$$Q_{p\cdot n} \equiv \sum_{j=1}^{M} \mathrm{e}^{p\cdot in\varphi_j}, p \in \{1,2,3,\ldots\}. \quad (9)$$

To obtain all the basis vectors of the subset $l = 4$, each composition in (8) must be combined by the ordered composition of the complex conjugate vectors, as shown in the following pattern:



(10)

This pattern gives fifteen ($p(4)[p(4)+1]/2 = 15$) different combinations:

$$\begin{array}{lll} Q_nQ_nQ_nQ_n|Q_n^*Q_n^*Q_n^*Q_n^* & Q_{2n}Q_nQ_n|Q_n^*Q_n^*Q_n^*Q_n^* & Q_{2n}Q_{2n}|Q_n^*Q_n^*Q_n^*Q_n^* \\ & Q_{2n}Q_nQ_n|Q_{2n}^*Q_n^*Q_n^* & Q_{2n}Q_{2n}|Q_{2n}^*Q_n^*Q_n^* \\ & & Q_{2n}Q_{2n}|Q_{2n}^*Q_{2n}^* \end{array}$$

$$\begin{array}{ll} Q_{3n}Q_n|Q_n^*Q_n^*Q_n^*Q_n^* & Q_{4n}|Q_n^*Q_n^*Q_n^*Q_n^* \\ Q_{3n}Q_n|Q_{2n}^*Q_n^*Q_n^* & Q_{4n}|Q_{2n}^*Q_n^*Q_n^* \\ Q_{3n}Q_n|Q_{2n}^*Q_{2n}^* & Q_{4n}|Q_{2n}^*Q_{2n}^* \\ Q_{3n}Q_n|Q_{3n}^*Q_n^* & Q_{4n}|Q_{3n}^*Q_n^* \\ & Q_{4n}|Q_{4n}^* \end{array} \quad (11)$$

The real part of each of these additional vectors (11), together with the basis $B_{\langle 6\rangle}$, makes the complete basis of the 8-particle azimuthal correlations $B_{\langle 8\rangle}$. This presentation of the composition of the flow vectors separated from the complex conjugate part by a vertical bar is inspired by the work of Ref. [14].

The corresponding multiplying integer functions $f^{(m,l)}$ might also be obtained via mathematical induction. These integer functions are given by

$$\begin{cases} (M-2l)\prod_{j=m+l+1}^{2m-1}(M-j), & \text{for} \quad l = \{0,\ldots,m-2\} \\ M-2l, & \text{for} \quad l = m-1 \\ 1, & \text{for} \quad l = m \end{cases}. \quad (12)$$

For example, in the case of the 8-particle azimuthal correlations ($m = 4$),

$$f^{(m=4,l)} = \begin{cases} M(M-5)(M-6)(M-7), & \text{for} \quad l=0 \\ (M-2)(M-6)(M-7), & \text{for} \quad l=1 \\ (M-4)(M-7), & \text{for} \quad l=2 \\ (M-6), & \text{for} \quad l=3 \\ 1, & \text{for} \quad l=4 \end{cases}. \quad (13)$$

We list the integer functions and basis vectors for each of the $2m$-particle azimuthal correlations ($m=1,2,\ldots,7$) in the third and fourth columns, respectively, of Tables 1−7.

Now, we have all the necessary quantities to form the system of linear equations to obtain all the unknown coefficients for each of the $2m$-particle azimuthal correlations:

$$\begin{array}{ccc} \left(\sum\limits_{j_1\neq\ldots\neq j_{2m}=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}+\ldots+\varphi_{j_m}-\varphi_{j_{m+1}}-\ldots-\varphi_{j_{2m}}\right)}\right)_{\text{event1}} & = & x_1\left(f_1^{(m,l)}\mathbf{e}_1\right)_{\text{event1}}+x_2\left(f_2^{(m,l)}\mathbf{e}_2\right)_{\text{event1}}+\ldots+x_d\left(f_d^{(m,l)}\mathbf{e}_d\right)_{\text{event1}} \\ \left(\sum\limits_{j_1\neq\ldots\neq j_{2m}=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}+\ldots+\varphi_{j_m}-\varphi_{j_{m+1}}-\ldots-\varphi_{j_{2m}}\right)}\right)_{\text{event2}} & = & x_1\left(f_1^{(m,l)}\mathbf{e}_1\right)_{\text{event2}}+x_2\left(f_2^{(m,l)}\mathbf{e}_2\right)_{\text{event2}}+\ldots+x_d\left(f_d^{(m,l)}\mathbf{e}_d\right)_{\text{event2}} \\ \vdots & = & \vdots \\ \left(\sum\limits_{j_1\neq\ldots\neq j_{2m}=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}+\ldots+\varphi_{j_m}-\varphi_{j_{m+1}}-\ldots-\varphi_{j_{2m}}\right)}\right)_{\text{event}d} & = & x_1\left(f_1^{(m,l)}\mathbf{e}_1\right)_{\text{event}d}+x_2\left(f_2^{(m,l)}\mathbf{e}_2\right)_{\text{event}d}+\ldots+x_d\left(f_d^{(m,l)}\mathbf{e}_d\right)_{\text{event}d} \end{array}, \quad (14)$$

To set a solvable system of Eq. (14), the multiplicity must be $M\geqslant 2m$ (otherwise, the system of equations cannot be formed). For example, the system of equations for the 2-particle azimuthal correlations is given by ($M\geqslant 2$)

$$\begin{array}{c} \left(\sum\limits_{j_1\neq j_2=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}-\varphi_{j_2}\right)}\right)_{\text{event1}} = x_1(M\cdot 1)_{\text{event1}}+x_2\left(1\cdot|Q_n|^2\right)_{\text{event1}} \\ \left(\sum\limits_{j_1\neq j_2=1}^{M} \mathrm{e}^{in\left(\varphi_{j_1}-\varphi_{j_2}\right)}\right)_{\text{event2}} = x_1(M\cdot 1)_{\text{event2}}+x_2\left(1\cdot|Q_n|^2\right)_{\text{event2}} \end{array} \quad (15)$$

Note that in the sum on the left-hand side of Eq. (15), a complex number and its conjugate are included; hence, the sum is reduced to a real number. To set the system of Eq. (15), we randomly choose two sets of angles: $\varphi_{j_1}=0.400906$, $\varphi_{j_2}=-2.84149$, $\varphi_{j_3}=1.98067$ (multiplicity $M=3$) for the first equation and $\varphi_{j_1}=-1.32161$, $\varphi_{j_2}=2.75646$, $\varphi_{j_3}=2.8089$, $\varphi_{j_4}=1.59479$, $\varphi_{j_5}=1.13565$, ($M=5$) for the second one. Then, the system of equations given by Eq. (15) for $n=2$ becomes

$$\begin{array}{c} -1.992180682386965 = x_1\cdot 3+x_2\cdot 1.007819317613035 \\ -2.80896273559164 = x_1\cdot 5+x_2\cdot 2.19103726440836 \end{array}. \quad (16)$$

Using the same sets of azimuthal angles for the Fourier harmonic $n=3$, the system of Eq. (15) becomes

$$\begin{array}{c} -2.502229604410126 = x_1\cdot 3+x_2\cdot 0.497770395589874 \\ 2.85297668985644 = x_1\cdot 5+x_2\cdot 7.85297668985644 \end{array}. \quad (17)$$

These two systems of linear equations are mutually different but both have the same solution: $x_1=-1$, $x_2=1$; therefore, we fill the second column of Table 1 with the corresponding x_i values.

The sets of azimuthal angles may be chosen randomly, as in the upper example. This should ensure that the calculated basis $B_{\langle 2m\rangle}$ does not have collinear vectors; otherwise, the linear system of equations is not solvable.

The calculated numbers that enter the system of Eqs. (16) and (17) should be given an appropriate number of significant digits for the obtained coefficients x_i to be true integers. There are no criteria to determine the appropriate number of significant digits. We can use the 'trial and error' method to find the required number. Otherwise, the obtained x_i will not be integers, and the rounding of x_i might be problematic in the case of a large basis $B_{\langle 2m\rangle}$. The limits of the application of this method depend on the number of significant digits by which the computer operates. We successfully apply the method up to the 14-th order of cumulants.

The two-particle azimuthal correlation formula can be formed by simply reading the corresponding values from Table 1:

Table 1. Coefficients, integer functions, and basis vectors for the calculation of the two-particle azimuthal correlations.

i	x_i	$f_i^{(m=1,l)}$	Basis vectors	l
1	−1	M	1	0
2	1	1	$Q_n\|Q_n^*$	1

$$\langle 2\rangle P_{M,2} = x_1 f_1^{(m=1,0)}1 + x_2 f_2^{(m=1,1)}\mathrm{Re}\left(Q_n Q_n^*\right) = -M + |Q_n|^2. \tag{18}$$

A similar table can be obtained for the four-particle azimuthal correlations using the solution of the corresponding linear system of equations.

This can also be easily rewritten as an appropriate formula by simply reading the corresponding values from Table 2:

$$\begin{aligned}\langle 4\rangle P_{M,4} =& x_1 f_1^{(m=2,0)}1 + x_2 f_2^{(m=2,1)}\mathrm{Re}(Q_n Q_n^*)\\ &+ x_3 f_3^{(m=2,2)}\mathrm{Re}(Q_n Q_n Q_n^* Q_n^*)+\\ &+ x_4 f_4^{(m=2,2)}\mathrm{Re}(Q_{2n} Q_n^* Q_n^*)\\ &+ x_5 f_5^{(m=2,2)}\mathrm{Re}(Q_{2n} Q_{2n}^*)\\ =&2M(M-3)-4(M-2)|Q_n|^2+|Q_n|^4\\ &-2\mathrm{Re}(Q_{2n}Q_n^*Q_n^*)+|Q_{2n}|^2 \quad .\end{aligned} \tag{19}$$

The appropriate tables for the higher order cumulants are given in the appendix.

Many of the basis vectors are complex valued numbers such as $Q_{2n}Q_n^*Q_n^*$, and one should take only their real part when writing the expression for the $2m$-particle azimuthal correlations. In fact, the analytical derivation reveals that in addition to these complex valued basis vectors, their complex conjugates ($Q_{2n}^*Q_nQ_n$) also participate in the corresponding basis. Because of symmetry, they always enter the expression for the $2m$-particle azimuthal correlations with the same coefficients and hence their imaginary parts cancel. This also causes the corresponding coefficients to be even numbers. For example, $x_i' f_i^{(m,l)} Q_{2n}Q_n^*Q_n^* + x_i' f_i^{(m,l)} Q_{2n}^*Q_nQ_n = 2x_i' f_i^{(m,l)}\mathrm{Re}(Q_{2n}Q_n^*Q_n^*)$ and therefore $x_i = 2x_i'$ is always an even number. Odd coefficients might appear only in front of basis vectors that are not accompanied by their complex conjugates, such as $|Q_n|^4$ and $|Q_{2n}|^2$, in Eq. (19).

The coefficients also have other interesting features that should be noticed. For example, all coefficients that correspond to the same subset of basis vectors (having the same index l) sum up to zero: (Table 3: $\sum_{i=6}^{11} x_i = 0$, $\sum_{i=3}^{5} x_i = 0$), (Table 4: $\sum_{i=12}^{26} x_i = 0$, and so on). The exceptions to this rule are the two coefficients at the beginning of each table that correspond to the basis vectors 1 and $|Q_n|^2$. The coefficient in front of the basis vector "1" is $(-1)^m m!$. One more interesting feature that may be used to check the correctness of the obtained expression for the $2m$-particle azimuthal correlations is the sum of the absolute values of all coefficients in a table. This sum reads as

$$\sum_{i=1}^{d}|x_i| = \left\{\begin{array}{ll} 2, & \text{for} \quad m=1\\ 10, & \text{for} \quad m=2\\ 96, & \text{for} \quad m=3\\ 1560, & \text{for} \quad m=4\\ 39120, & \text{for} \quad m=5\\ 1409040, & \text{for} \quad m=6\\ 69048000, & \text{for} \quad m=7\end{array}\right\} = m!e\Gamma(m+1,1), \tag{20}$$

where d is the dimension of the vector space, e is Euler's number, and $\Gamma(a,x) = \int_x^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$ is the incomplete gamma function of x with parameter a.

We also apply this method of derivation to the case of the weighted Q-vector evaluated in the harmonic n:

Table 2. Coefficients, integer functions, and basis vectors for the calculation of the four-particle azimuthal correlations.

i	x_i	$f_i^{(m=2,l)}$	Basis vectors	l
1	2	$M(M-3)$	1	0
2	−4	$(M-2)$	$Q_n\|Q_n^*$	1
3	1	1	$Q_nQ_n\|Q_n^*Q_n^*$	2
4	−2	1	$Q_{2n}\|Q_n^*Q_n^*$	2
5	1	1	$Q_{2n}\|Q_{2n}^*$	2

Table 3. Coefficients, integer functions, and basis vectors for the calculation of the six-particle azimuthal correlations.

i	x_i	$f_i^{(m=3,l)}$	Basis vectors	l
1	−6	$M(M-4)(M-5)$	1	0
2	18	$(M-2)(M-5)$	$Q_n\|Q_n^*$	1
3	−9	$(M-4)$	$Q_nQ_n\|Q_n^*Q_n^*$	2
4	18	$(M-4)$	$Q_{2n}\|Q_n^*Q_n^*$	2
5	−9	$(M-4)$	$Q_{2n}\|Q_{2n}^*$	2
6	1	1	$Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*$	3
7	−6	1	$Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*$	3
8	9	1	$Q_{2n}Q_n\|Q_n^*Q_{2n}^*$	3
9	4	1	$Q_{3n}\|Q_n^*Q_n^*Q_n^*$	3
10	−12	1	$Q_{3n}\|Q_n^*Q_{2n}^*$	3
11	4	1	$Q_{3n}\|Q_{3n}^*$	3

Table 4. Coefficients, integer functions, and basis vectors for the calculation of the eight-particle azimuthal correlations.

i	x_i	$f_i^{(m=4,l)}$	Basis vectors	l
1	24	$M(M-5)(M-6)(M-7)$	1	0
2	−96	$(M-2)(M-6)(M-7)$	$Q_n\|Q_n^*$	1
3	72	$(M-4)(M-7)$	$Q_nQ_n\|Q_n^*Q_n^*$	2
4	−144	$(M-4)(M-7)$	$Q_{2n}\|Q_n^*Q_n^*$	2
5	72	$(M-4)(M-7)$	$Q_{2n}\|Q_{2n}^*$	2
6	−16	$(M-6)$	$Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*$	3
7	96	$(M-6)$	$Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*$	3
8	−144	$(M-6)$	$Q_{2n}Q_n\|Q_n^*Q_{2n}^*$	3
9	−64	$(M-6)$	$Q_{3n}\|Q_n^*Q_n^*Q_n^*$	3
10	192	$(M-6)$	$Q_{3n}\|Q_n^*Q_{2n}^*$	3
11	−64	$(M-6)$	$Q_{3n}\|Q_{3n}^*$	3
12	1	1	$Q_nQ_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
13	−12	1	$Q_{2n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
14	36	1	$Q_{2n}Q_nQ_n\|Q_n^*Q_n^*Q_{2n}^*$	4
15	6	1	$Q_{2n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
16	−36	1	$Q_{2n}Q_{2n}\|Q_n^*Q_n^*Q_{2n}^*$	4
17	9	1	$Q_{2n}Q_{2n}\|Q_{2n}^*Q_{2n}^*$	4
18	16	1	$Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
19	−96	1	$Q_{3n}Q_n\|Q_n^*Q_n^*Q_{2n}^*$	4
20	48	1	$Q_{3n}Q_n\|Q_{2n}^*Q_{2n}^*$	4
21	64	1	$Q_{3n}Q_n\|Q_n^*Q_{3n}^*$	4
22	−12	1	$Q_{4n}\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
23	72	1	$Q_{4n}\|Q_n^*Q_n^*Q_{2n}^*$	4
24	−36	1	$Q_{4n}\|Q_{2n}^*Q_{2n}^*$	4
25	−96	1	$Q_{4n}\|Q_n^*Q_{3n}^*$	4
26	36	1	$Q_{4n}\|Q_{4n}^*$	4

$$Q_{n,q}=\sum_{j=1}^{M}\omega_j^q e^{in\varphi_j}, \tag{21}$$

where w_j is the weight of the j-th particle [17]. The derivation in this case is more complicated primarily because there are more basis vectors involved. However, the obtained coefficients in front of the basis vectors have simple features:

$$\sum_{i=1}^{d'} x_i = 0, \tag{22}$$

$$\sum_{i=1}^{d'} |x_i| = (2m)!. \tag{23}$$

(The dimension of the vector space in this case is not equal to the previous one, that is, $d' \neq d$.)

Finally, with expressions for the $2m$-particle azimuthal correlations $\langle 2m\rangle$, we can calculate the weighted average over all events $\langle\langle 2m\rangle\rangle$ (given by Eq. (1) in Ref. [18]). Then, we can use the recurrence relation to calculate the cumulants of any order by knowing all the cumulants of lower orders [18]:

$$c_n\{2k\}=\langle\langle 2k\rangle\rangle-\sum_{m=1}^{k-1}\begin{pmatrix}k\\m\end{pmatrix}\begin{pmatrix}k-1\\m\end{pmatrix}\langle\langle 2m\rangle\rangle c_n\{2k-2m\}. \tag{24}$$

The cumulant based flow harmonics $v_n\{2k\}$ (k = 1, 2…) can be calculated using the following:

$$v_n\{2k\}=\sqrt[2k]{a_{2k}^{-1}c_n\{2k\}}, \tag{25}$$

where the coefficients a_{2k} are obtainable via the recursion relation

$$a_{2k}=1-\sum_{m=1}^{k-1}\begin{pmatrix}k\\m\end{pmatrix}\begin{pmatrix}k-1\\m\end{pmatrix}a_{2k-2m}, \text{with}: a_2=1, \tag{26}$$

which enables easy calculation of high-order $v_n\{2k\}$ using any commercial program for calculation. For example, $v_n\{2\}=\sqrt[2]{c_n\{2\}}$, $v_n\{8\}=\sqrt[8]{[-33]^{-1}c_n\{8\}}$,

$$v_n\{16\}=\sqrt[16]{[-10643745]^{-1}c_n\{16\}},$$

$$v_n\{26\}=\sqrt[26]{[24730000147369440]^{-1}c_n\{26\}},$$

$$v_n\{38\}=\sqrt[38]{[706967553323274026408967101760]^{-1}c_n\{38\}}.$$

V. DEMONSTRATION OF THE METHOD USING A TOY MODEL

The experimental values of $v_2\{2k\}$ enable us to determine the central moments of the v_2 distribution. As a way of obtaining the lowest central moments of the v_2 distribution as the variances σ_{20}^2 and σ_{02}^2, skewness s_{30}, and co-skewness s_{12} [15, 16], experimental values of at least four different cumulants $v_2\{2\}$, $v_2\{4\}$, $v_2\{6\}$, and $v_2\{8\}$ are required. However, the centrality dependence of the hydrodynamic probe $h=(v_2\{6\}-v_2\{8\})/(v_2\{4\}-v_2\{6\})$ obtained in the experiment [16] indicates a nonzero value of the kurtosis κ_{40}, which is the fourth central moment of the v_2 distribution. Therefore, to obtain the additional fourth central moments, κ_{40}, κ_{22}, and κ_{04} [16], experimental values of at least three more cumulants of higher order, $v_2\{10\}$, $v_2\{12\}$, and $v_2\{14\}$, are required.

Therefore, to show the validity of the above method, the obtained expressions for cumulants up to the 14-th order are calculated with the azimuthal angles simulated using toy models. For each event of a given v_2, a simple distribution $p(\varphi)=\frac{1}{2\pi}(1+2v_2\cos(2\varphi))$ is used to generate the particle azimuthal angle. We set the input value $v_2=0.15$ with no event-by-event flow fluctuations. Thus, we expect that all $v_2\{2k\}$ are equal to the input value. Fig. 1 shows that this is indeed what is observed, because there is excellent agreement between the cumulant based $v_2\{2k\}$ estimations and the input v_2 value of 0.15. Owing to a strong correlation between different $\langle\langle 2m\rangle\rangle$ [18], the statistical uncertainties on $v_2\{2k\}$ slowly increase with increasing k. This toy model therefore validates the method.

However, from this validation, the sensitivity to the flow fluctuations of the higher-order cumulants cannot be verified. Therefore, we apply the second toy model, where the initial eccentricity ε_2 distribution is simulated using the elliptic power distribution:

$$\frac{\mathrm{d}N}{\mathrm{d}\varepsilon_2}=\frac{2\alpha}{\pi}(1-\varepsilon_0^2)^{\alpha+1/2}\varepsilon_2(1-\varepsilon_2^2)^{\alpha-1}\int_0^{\pi}\frac{1}{(1-\varepsilon_0\varepsilon_2\cos\varphi)^{2\alpha+1}}\mathrm{d}\phi, \tag{27}$$

where α and ε_0 are the power and ellipticity parameters, respectively, which take different values, obtained using the Glauber model for 5.02 GeV Pb-Pb collisions, depending on the centrality [19]. The scaling factor κ_2 between the elliptic flow and initial eccentricity, $v_2=\kappa_2\varepsilon_2$, is chosen to imitate the centrality dependence of the elliptic flow v_2 measured in Ref. [16]. For each event of a given v_2, a simple distribution $1+2v_2\cos(2\varphi)$ is used to

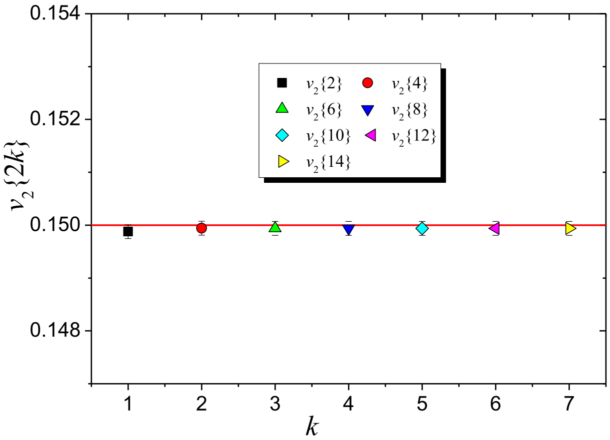


Fig. 1. (color online) Cumulant based estimations of the elliptic flow v_2 from toy Monte Carlo simulations. The input value of v_2 is 0.15. All cumulants retrieve this input value with high precision.

generate the particle azimuthal angle. The integration in Eq. (27) can be carried out analytically to give [20]

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}\varepsilon_2}=&2\alpha(1-\varepsilon_0^2)^{\alpha+1/2}\varepsilon_2\frac{(1-\varepsilon_2^2)^{\alpha-1}}{(1-\varepsilon_0\varepsilon_2)^{2\alpha+1}}\\&\times{}_2F_1\left(\frac{1}{2},2\alpha+1;1;\frac{2\varepsilon_0\varepsilon_2}{\varepsilon_0\varepsilon_2-1}\right).\end{aligned} \tag{28}$$

However, the ROOT version [21] of the hypergeometric function in Eq. (28) is not defined everywhere in the interval (0, 1) of ε_2. Fig. 2 shows the parametric area of definition of the ROOT version of the hypergeometric function.

As shown, only the centrality range 0−15% of the 5.02 GeV Pb-Pb collisions can be well simulated using this hypergeometric function. We solve this inconvenience by applying the Pfaff transformation,

$${}_2F_1(a,b;c;z)=(1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right), \tag{29}$$

which gives the following eccentricity distribution, well defined for all parameter values:

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}\varepsilon_2}=&2\alpha(1-\varepsilon_0^2)^{\alpha+1/2}\varepsilon_2\frac{(1-\varepsilon_2^2)^{\alpha-1}}{(1+\varepsilon_0\varepsilon_2)^{2\alpha+1}}\\&\times{}_2F_1\left(\frac{1}{2},2\alpha+1;1;\frac{2\varepsilon_0\varepsilon_2}{1+\varepsilon_0\varepsilon_2}\right).\end{aligned} \tag{30}$$

For each centrality, approximately 1.5×10^7 events are simulated using the above toy model. Figure 3 shows the $v_2\{2k\}$ values ($k=1,\ldots,7$) calculated based on the obtained expressions for the corresponding $\langle 2m\rangle$ correla-

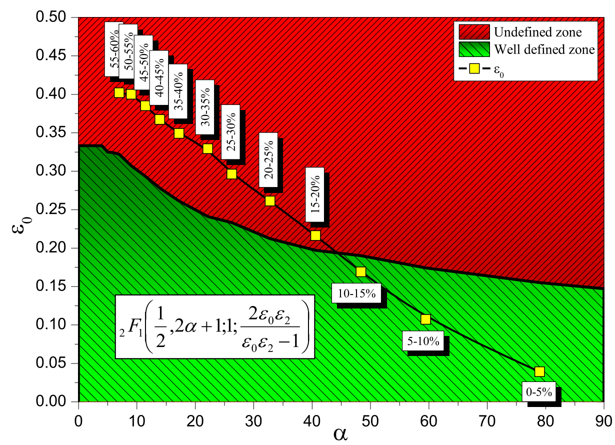


Fig. 2. (color online) Parameter values (α, ε_0) of the elliptic power distribution for 5.02 GeV Pb-Pb collisions calculated in the Glauber model for different centralities, and the area of definition of the ROOT version of the hypergeometric function ${}_2F_1\left(\frac{1}{2},2\alpha+1;1;\frac{2\varepsilon_0\varepsilon_2}{\varepsilon_0\varepsilon_2-1}\right)$.

tions as a function of centrality. A gap between $v_2\{2\}$ and the higher-order cumulant values $v_2\{2k\}$ ($k = 2,\ldots,7$) is present. This is due to flow fluctuations, which relate the higher-order cumulants based $v_2\{2k\}$ and the variance σ_v of the v_2 distribution: $v_2\{2\}^2 \approx v_2\{2k\}^2 + 2\sigma^2{}_v$ for $k > 1$ [15]. As shown from the experimental data [16], flow fluctuations become larger moving toward peripheral collisions. The elliptic flow v_2 values are reproduced using the expressions for $\langle 2m\rangle$ developed in this paper.

Because the fluctuations in the initial state are not Gaussian, the $v_2\{2k\}$ values for $k > 1$ will not be the same. This will produce a splitting between different $v_2\{2k\}$ values, and they will be ordered as $v_2\{2k\} > v_2\{2(k+1)\}$ for any $k > 1$. In Fig. 3, the ordering and fine splitting between the $v_2\{2k\}$ values ($k = 2,\ldots,7$) are not clearly visible. To ensure that the splitting between the cumulants of different orders are explicitly visible, we show the relative differences $(v_2\{2k\} - v_2\{14\})/v_2\{14\}$ ($k = 1,\ldots,6$) as a function of centrality in Fig. 4, where different symbols denote different cumulant orders, which are calculated from the simulated data. The corresponding input values, obtained directly by applying the elliptic power distributions, are represented by spline interpolation lines. In Fig. 4, the ordering can be clearly observed, as well as the fine splitting between cumulants of different orders. The relative difference between the cumulants decreases by approximately one order of magnitude for each increment of the order k. The excellent agreement between the lines and symbols proves the correctness of the obtained expressions for the $2m$-particle azimuthal correlations $\langle 2m\rangle$.

VI. CONCLUSIONS

In this paper, we present a method, based on mathematical induction, that allows us to calculate high-order $v_2\{2k\}$ values from Q-cumulants in a relatively straightforward way. The analytical expressions for the high $2m$-particle correlations $\langle 10\rangle$, $\langle 12\rangle$, and $\langle 14\rangle$ are obtained for the first time and given here in the form of table values. The validity of the proposed method is confirmed via an elliptic flow simulation with a toy model using the elliptic power distribution. We transform the hypergeometric function involved in the elliptic power distribution by applying Pfaff transformation and enable its calculation in the ROOT program for all parameter values of the v_2 distribution. The ability to calculate high-order $v_2\{2k\}$ values allows for the possibility of studying the fine details of the v_2 distribution by extracting its skewness and higher central moments. Theoreticians can tune initial states in their hydrodynamic models to reconstruct the measured central moments. This can place stringent constraints on the initial geometry used in hydrodynamic calculations of the collective dynamics of QGP in high energy nuclear collisions.

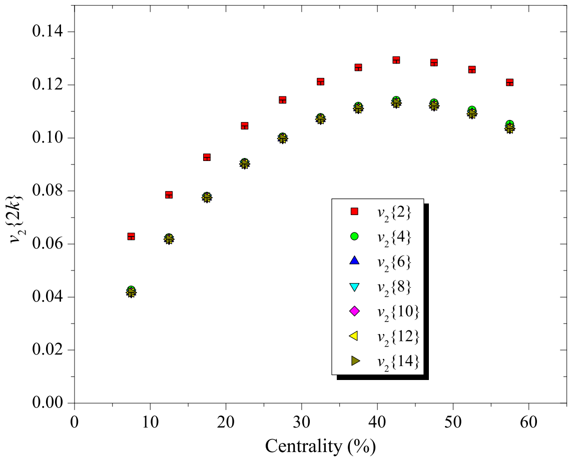


Fig. 3. (color online) $v_2\{2k\}$ values ($k = 1,\ldots,7$) as a function of centrality calculated from the data simulated using the elliptic power distribution toy model. The statistical uncertainties are negligible compared to the marker size.

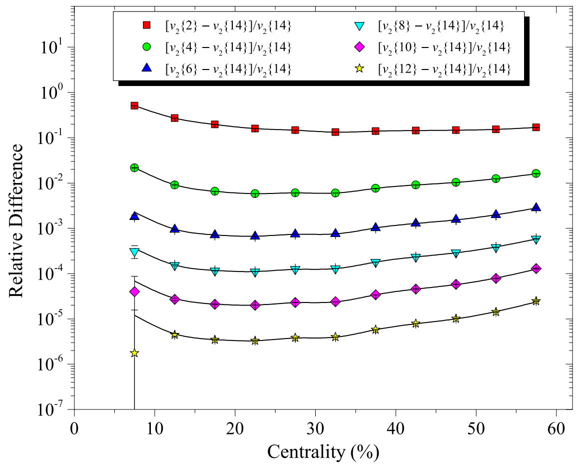


Fig. 4. (color online) Relative differences $(v_2\{2k\} - v_2\{14\})/v_2\{14\}$ ($k = 1,\ldots,6$) as a function of centrality. The input values, obtained directly by applying the elliptic power distributions, are represented by spline interpolation lines.

APPENDIX

This appendix provides tables listing (Tables 5–7), for different orders of Q-cumulants, the coefficients, integer functions, and combinations of basis vectors needed to construct expressions for the $2m$-particle azimuthal correlations $\langle 2m\rangle$.

Table 5. Coefficients, integer functions, and basis vectors for the calculation of the ten-particle azimuthal correlations.

i	x_i	$f_i^{(m=5,l)}$	Basis vectors	l
1	−120	$M(M-6)(M-7)(M-8)(M-9)$	1	0
2	600	$(M-2)(M-7)(M-8)(M-9)$	$Q_n\|Q_n^*$	1
3	−600	$(M-4)(M-8)(M-9)$	$Q_nQ_n\|Q_n^*Q_n^*$	2
4	1200	$(M-4)(M-8)(M-9)$	$Q_{2n}\|Q_n^*Q_n^*$	2
5	−600	$(M-4)(M-8)(M-9)$	$Q_{2n}\|Q_{2n}^*$	2
6	200	$(M-6)(M-9)$	$Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*$	3
7	−1200	$(M-6)(M-9)$	$Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*$	3
8	1800	$(M-6)(M-9)$	$Q_{2n}Q_n\|Q_n^*Q_{2n}^*$	3
9	800	$(M-6)(M-9)$	$Q_{3n}\|Q_n^*Q_n^*Q_n^*$	3
10	−2400	$(M-6)(M-9)$	$Q_{3n}\|Q_n^*Q_{2n}^*$	3
11	800	$(M-6)(M-9)$	$Q_{3n}\|Q_{3n}^*$	3
12	−25	$(M-8)$	$Q_nQ_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
13	300	$(M-8)$	$Q_{2n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
14	−900	$(M-8)$	$Q_{2n}Q_nQ_n\|Q_n^*Q_n^*Q_{2n}^*$	4
15	−150	$(M-8)$	$Q_{2n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
16	900	$(M-8)$	$Q_{2n}Q_{2n}\|Q_n^*Q_n^*Q_{2n}^*$	4
17	−225	$(M-8)$	$Q_{2n}Q_{2n}\|Q_{2n}^*Q_{2n}^*$	4
18	−400	$(M-8)$	$Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
19	2400	$(M-8)$	$Q_{3n}Q_n\|Q_n^*Q_n^*Q_{2n}^*$	4
20	−1200	$(M-8)$	$Q_{3n}Q_n\|Q_{2n}^*Q_{2n}^*$	4
21	−1600	$(M-8)$	$Q_{3n}Q_n\|Q_n^*Q_{3n}^*$	4
22	300	$(M-8)$	$Q_{4n}\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
23	−1800	$(M-8)$	$Q_{4n}\|Q_n^*Q_n^*Q_{2n}^*$	4
24	900	$(M-8)$	$Q_{4n}\|Q_{2n}^*Q_{2n}^*$	4
25	2400	$(M-8)$	$Q_{4n}\|Q_n^*Q_{3n}^*$	4
26	−900	$(M-8)$	$Q_{4n}\|Q_{4n}^*$	4
27	1	1	$Q_nQ_nQ_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
28	−20	1	$Q_{2n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
29	100	1	$Q_{2n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*$	5
30	30	1	$Q_{2n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
31	−300	1	$Q_{2n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*$	5
32	225	1	$Q_{2n}Q_{2n}Q_n\|Q_n^*Q_{2n}^*Q_{2n}^*$	5
33	40	1	$Q_{3n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
34	−400	1	$Q_{3n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*$	5
35	600	1	$Q_{3n}Q_nQ_n\|Q_n^*Q_{2n}^*Q_{2n}^*$	5
36	400	1	$Q_{3n}Q_nQ_n\|Q_n^*Q_n^*Q_{3n}^*$	5
37	−40	1	$Q_{3n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
38	400	1	$Q_{3n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_{2n}^*$	5
39	−600	1	$Q_{3n}Q_{2n}\|Q_n^*Q_{2n}^*Q_{2n}^*$	5
40	−800	1	$Q_{3n}Q_{2n}\|Q_n^*Q_n^*Q_{3n}^*$	5
41	400	1	$Q_{3n}Q_{2n}\|Q_{2n}^*Q_{3n}^*$	5

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Table 5-continued from previous page

i	x_i	$f_i^{(m=5,l)}$	Basis vectors	l
42	−60	1	$Q_{4n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
43	600	1	$Q_{4n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*$	5
44	−900	1	$Q_{4n}Q_n\|Q_n^*Q_{2n}^*Q_{2n}^*$	5
45	−1200	1	$Q_{4n}Q_n\|Q_n^*Q_n^*Q_{3n}^*$	5
46	1200	1	$Q_{4n}Q_n\|Q_{2n}^*Q_{3n}^*$	5
47	900	1	$Q_{4n}Q_n\|Q_n^*Q_{4n}^*$	5
48	48	1	$Q_{5n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	5
49	−480	1	$Q_{5n}\|Q_n^*Q_n^*Q_n^*Q_{2n}^*$	5
50	720	1	$Q_{5n}\|Q_n^*Q_{2n}^*Q_{2n}^*$	5
51	960	1	$Q_{5n}\|Q_n^*Q_n^*Q_{3n}^*$	5
52	−960	1	$Q_{5n}\|Q_{2n}^*Q_{3n}^*$	5
53	−1440	1	$Q_{5n}\|Q_n^*Q_{4n}^*$	5
54	576	1	$Q_{5n}\|Q_{5n}^*$	5

Table 6. Coefficients, integer functions, and basis vectors for the calculation of the twelve-particle azimuthal correlations.

i	x_i	$f_i^{(m=6,l)}$	Basis vectors	l
1	720	$M(M-7)(M-8)(M-9)(M-10)(M-11)$	1	0
2	−4320	$(M-2)(M-8)(M-9)(M-10)(M-11)$	$Q_n\|Q_n^*$	1
3	5400	$(M-4)(M-9)(M-10)(M-11)$	$Q_nQ_n\|Q_n^*Q_n^*$	2
4	−10800	$(M-4)(M-9)(M-10)(M-11)$	$Q_{2n}\|Q_n^*Q_n^*$	2
5	5400	$(M-4)(M-9)(M-10)(M-11)$	$Q_{2n}\|Q_{2n}^*$	2
6	−2400	$(M-6)(M-10)(M-11)$	$Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*$	3
7	14400	$(M-6)(M-10)(M-11)$	$Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*$	3
8	−21600	$(M-6)(M-10)(M-11)$	$Q_{2n}Q_n\|Q_n^*Q_{2n}^*$	3
9	−9600	$(M-6)(M-10)(M-11)$	$Q_{3n}\|Q_n^*Q_n^*Q_n^*$	3
10	28800	$(M-6)(M-10)(M-11)$	$Q_{3n}\|Q_n^*Q_{2n}^*$	3
11	−9600	$(M-6)(M-10)(M-11)$	$Q_{3n}\|Q_{3n}^*$	3
12	450	$(M-8)(M-11)$	$Q_nQ_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
13	−5400	$(M-8)(M-11)$	$Q_{2n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
14	16200	$(M-8)(M-11)$	$Q_{2n}Q_nQ_n\|Q_n^*Q_n^*Q_{2n}^*$	4
15	2700	$(M-8)(M-11)$	$Q_{2n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
16	−16200	$(M-8)(M-11)$	$Q_{2n}Q_{2n}\|Q_n^*Q_n^*Q_{2n}^*$	4
17	4050	$(M-8)(M-11)$	$Q_{2n}Q_{2n}\|Q_{2n}^*Q_{2n}^*$	4
18	7200	$(M-8)(M-11)$	$Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
19	−43200	$(M-8)(M-11)$	$Q_{3n}Q_n\|Q_n^*Q_n^*Q_{2n}^*$	4
20	21600	$(M-8)(M-11)$	$Q_{3n}Q_n\|Q_{2n}^*Q_{2n}^*$	4
21	28800	$(M-8)(M-11)$	$Q_{3n}Q_n\|Q_n^*Q_{3n}^*$	4
22	−5400	$(M-8)(M-11)$	$Q_{4n}\|Q_n^*Q_n^*Q_n^*Q_n^*$	4
23	32400	$(M-8)(M-11)$	$Q_{4n}\|Q_n^*Q_n^*Q_{2n}^*$	4
24	−16200	$(M-8)(M-11)$	$Q_{4n}\|Q_{2n}^*Q_{2n}^*$	4
25	−43200	$(M-8)(M-11)$	$Q_{4n}\|Q_n^*Q_{3n}^*$	4

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Table 6-continued from previous pagee

i	x_i	$f_i^{(m=6,l)}$	Basis vectors	l
26	16200	$(M-8)(M-11)$	$Q_{4n}\|Q^*_{4n}$	4
27	−36	$(M-10)$	$Q_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
28	720	$(M-10)$	$Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
29	−3600	$(M-10)$	$Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
30	−1080	$(M-10)$	$Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
31	10800	$(M-10)$	$Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
32	−8100	$(M-10)$	$Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
33	−1440	$(M-10)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
34	14400	$(M-10)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
35	−21600	$(M-10)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
36	−14400	$(M-10)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_nQ^*_{3n}$	5
37	1440	$(M-10)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
38	−14400	$(M-10)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
39	21600	$(M-10)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
40	28800	$(M-10)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_nQ^*_{3n}$	5
41	−14400	$(M-10)$	$Q_{3n}Q_{2n}\|Q^*_{2n}Q^*_{3n}$	5
42	2160	$(M-10)$	$Q_{4n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
43	−21600	$(M-10)$	$Q_{4n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
44	32400	$(M-10)$	$Q_{4n}Q_n\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
45	43200	$(M-10)$	$Q_{4n}Q_n\|Q^*_nQ^*_nQ^*_{3n}$	5
46	−43200	$(M-10)$	$Q_{4n}Q_n\|Q^*_{2n}Q^*_{3n}$	5
47	−32400	$(M-10)$	$Q_{4n}Q_n\|Q^*_nQ^*_{4n}$	5
48	−1728	$(M-10)$	$Q_{5n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
49	17280	$(M-10)$	$Q_{5n}\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
50	−25920	$(M-10)$	$Q_{5n}\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
51	−34560	$(M-10)$	$Q_{5n}\|Q^*_nQ^*_nQ^*_{3n}$	5
52	34560	$(M-10)$	$Q_{5n}\|Q^*_{2n}Q^*_{3n}$	5
53	51840	$(M-10)$	$Q_{5n}\|Q^*_nQ^*_{4n}$	5
54	−20736	$(M-10)$	$Q_{5n}\|Q^*_{5n}$	5
55	1	1	$Q_nQ_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
56	−30	1	$Q_{2n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
57	225	1	$Q_{2n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
58	90	1	$Q_{2n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
59	−1350	1	$Q_{2n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
60	2025	1	$Q_{2n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	6
61	−30	1	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
62	450	1	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
63	−1350	1	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	6
64	225	1	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_{2n}Q^*_{2n}Q^*_{2n}$	6
65	80	1	$Q_{3n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
66	−1200	1	$Q_{3n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6

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Table 6-continued from previous page

i	x_i	$f_i^{(m=6,l)}$	Basis vectors	l
67	3600	1	$Q_{3n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
68	−1200	1	$Q_{3n}Q_nQ_nQ_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
69	1600	1	$Q_{3n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
70	−240	1	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
71	3600	1	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
72	−10800	1	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
73	3600	1	$Q_{3n}Q_{2n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
74	−9600	1	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
75	14400	1	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
76	80	1	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
77	−1200	1	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
78	3600	1	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
79	−1200	1	$Q_{3n}Q_{3n}\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
80	3200	1	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
81	−9600	1	$Q_{3n}Q_{3n}\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
82	1600	1	$Q_{3n}Q_{3n}\|Q_{3n}^*Q_{3n}^*$	6
83	−180	1	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
84	2700	1	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
85	−8100	1	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
86	2700	1	$Q_{4n}Q_nQ_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
87	−7200	1	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
88	21600	1	$Q_{4n}Q_nQ_n\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
89	−7200	1	$Q_{4n}Q_nQ_n\|Q_{3n}^*Q_{3n}^*$	6
90	8100	1	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_{4n}^*$	6
91	180	1	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
92	−2700	1	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
93	8100	1	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
94	−2700	1	$Q_{4n}Q_{2n}\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
95	7200	1	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
96	−21600	1	$Q_{4n}Q_{2n}\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
97	7200	1	$Q_{4n}Q_{2n}\|Q_{3n}^*Q_{3n}^*$	6
98	−16200	1	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_{4n}^*$	6
99	8100	1	$Q_{4n}Q_{2n}\|Q_{2n}^*Q_{4n}^*$	6
100	288	1	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
101	−4320	1	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
102	12960	1	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
103	−4320	1	$Q_{5n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
104	11520	1	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
105	−34560	1	$Q_{5n}Q_n\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
106	11520	1	$Q_{5n}Q_n\|Q_{3n}^*Q_{3n}^*$	6
107	−25920	1	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_{4n}^*$	6

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Table 6-continued from previous page

i	x_i	$f_i^{(m=6,l)}$	Basis vectors	l
108	25920	1	$Q_{5n}Q_n\|Q^*_{2n}Q^*_{4n}$	6
109	20736	1	$Q_{5n}Q_n\|Q^*_nQ^*_{5n}$	6
110	-240	1	$Q_{6n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
111	3600	1	$Q_{6n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
112	−10800	1	$Q_{6n}\|Q^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	6
113	3600	1	$Q_{6n}\|Q^*_{2n}Q^*_{2n}Q^*_{2n}$	6
114	−9600	1	$Q_{6n}\|Q^*_nQ^*_nQ^*_nQ^*_{3n}$	6
115	28800	1	$Q_{6n}\|Q^*_nQ^*_{2n}Q^*_{3n}$	6
116	−9600	1	$Q_{6n}\|Q^*_{3n}Q^*_{3n}$	6
117	21600	1	$Q_{6n}\|Q^*_nQ^*_nQ^*_{4n}$	6
118	−21600	1	$Q_{6n}\|Q^*_{2n}Q^*_{4n}$	6
119	−34560	1	$Q_{6n}\|Q^*_nQ^*_{5n}$	6
120	14400	1	$Q_{6n}\|Q^*_{6n}$	6

Table 7. Coefficients, integer functions, and basis vectors for the calculation of the fourteen-particle azimuthal correlations.

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
1	−5040	$M(M-8)(M-9)(M-10)(M-11)(M-12)(M-13)$	1	0
2	35280	$(M-2)(M-9)(M-10)(M-11)(M-12)(M-13)$	$Q_n\|Q^*_n$	1
3	−52920	$(M-4)(M-10)(M-11)(M-12)(M-13)$	$Q_nQ_n\|Q^*_nQ^*_n$	2
4	105840	$(M-4)(M-10)(M-11)(M-12)(M-13)$	$Q_{2n}\|Q^*_nQ^*_n$	2
5	−52920	$(M-4)(M-10)(M-11)(M-12)(M-13)$	$Q_{2n}\|Q^*_{2n}$	2
6	29400	$(M-6)(M-11)(M-12)(M-13)$	$Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_n$	3
7	−176400	$(M-6)(M-11)(M-12)(M-13)$	$Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_n$	3
8	264600	$(M-6)(M-11)(M-12)(M-13)$	$Q_{2n}Q_n\|Q^*_nQ^*_{2n}$	3
9	117600	$(M-6)(M-11)(M-12)(M-13)$	$Q_{3n}\|Q^*_nQ^*_nQ^*_n$	3
10	−352800	$(M-6)(M-11)(M-12)(M-13)$	$Q_{3n}\|Q^*_nQ^*_{2n}$	3
11	117600	$(M-6)(M-11)(M-12)(M-13)$	$Q_{3n}\|Q^*_{3n}$	3
12	−7350	$(M-8)(M-12)(M-13)$	$Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_n$	4
13	88200	$(M-8)(M-12)(M-13)$	$Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_n$	4
14	−264600	$(M-8)(M-12)(M-13)$	$Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_{2n}$	4
15	−44100	$(M-8)(M-12)(M-13)$	$Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_n$	4
16	264600	$(M-8)(M-12)(M-13)$	$Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_{2n}$	4
17	−66150	$(M-8)(M-12)(M-13)$	$Q_{2n}Q_{2n}\|Q^*_{2n}Q^*_{2n}$	4
18	−117600	$(M-8)(M-12)(M-13)$	$Q_{3n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_n$	4
19	705600	$(M-8)(M-12)(M-13)$	$Q_{3n}Q_n\|Q^*_nQ^*_nQ^*_{2n}$	4
20	−352800	$(M-8)(M-12)(M-13)$	$Q_{3n}Q_n\|Q^*_{2n}Q^*_{2n}$	4
21	−470400	$(M-8)(M-12)(M-13)$	$Q_{3n}Q_n\|Q^*_nQ^*_{3n}$	4
22	88200	$(M-8)(M-12)(M-13)$	$Q_{4n}\|Q^*_nQ^*_nQ^*_nQ^*_n$	4
23	−529200	$(M-8)(M-12)(M-13)$	$Q_{4n}\|Q^*_nQ^*_nQ^*_{2n}$	4
24	264600	$(M-8)(M-12)(M-13)$	$Q_{4n}\|Q^*_{2n}Q^*_{2n}$	4
25	705600	$(M-8)(M-12)(M-13)$	$Q_{4n}\|Q^*_nQ^*_{3n}$	4

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Table 7-continued from previous page

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
26	−264600	$(M-8)(M-12)(M-13)$	$Q_{4n}\|Q^*_{4n}$	4
27	882	$(M-10)(M-13)$	$Q_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
28	−17640	$(M-10)(M-13)$	$Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
29	88200	$(M-10)(M-13)$	$Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
30	26460	$(M-10)(M-13)$	$Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
31	−264600	$(M-10)(M-13)$	$Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
32	198450	$(M-10)(M-13)$	$Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
33	35280	$(M-10)(M-13)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
34	−352800	$(M-10)(M-13)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
35	529200	$(M-10)(M-13)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
36	352800	$(M-10)(M-13)$	$Q_{3n}Q_nQ_n\|Q^*_nQ^*_nQ^*_{3n}$	5
37	−35280	$(M-10)(M-13)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
38	352800	$(M-10)(M-13)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
39	−529200	$(M-10)(M-13)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
40	−705600	$(M-10)(M-13)$	$Q_{3n}Q_{2n}\|Q^*_nQ^*_nQ^*_{3n}$	5
41	352800	$(M-10)(M-13)$	$Q_{3n}Q_{2n}\|Q^*_{2n}Q^*_{3n}$	5
42	−52920	$(M-10)(M-13)$	$Q_{4n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
43	529200	$(M-10)(M-13)$	$Q_{4n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
44	−793800	$(M-10)(M-13)$	$Q_{4n}Q_n\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
45	−1058400	$(M-10)(M-13)$	$Q_{4n}Q_n\|Q^*_nQ^*_nQ^*_{3n}$	5
46	1058400	$(M-10)(M-13)$	$Q_{4n}Q_n\|Q^*_{2n}Q^*_{3n}$	5
47	793800	$(M-10)(M-13)$	$Q_{4n}Q_n\|Q^*_nQ^*_{4n}$	5
48	42336	$(M-10)(M-13)$	$Q_{5n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	5
49	−423360	$(M-10)(M-13)$	$Q_{5n}\|Q^*_nQ^*_nQ^*_nQ^*_{2n}$	5
50	635040	$(M-10)(M-13)$	$Q_{5n}\|Q^*_nQ^*_{2n}Q^*_{2n}$	5
51	846720	$(M-10)(M-13)$	$Q_{5n}\|Q^*_nQ^*_nQ^*_{3n}$	5
52	−846720	$(M-10)(M-13)$	$Q_{5n}\|Q^*_{2n}Q^*_{3n}$	5
53	−1270080	$(M-10)(M-13)$	$Q_{5n}\|Q^*_nQ^*_{4n}$	5
54	508032	$(M-10)(M-13)$	$Q_{5n}\|Q^*_{5n}$	5
55	−49	$(M-12)$	$Q_nQ_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
56	1470	$(M-12)$	$Q_{2n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
57	−11025	$(M-12)$	$Q_{2n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
58	−4410	$(M-12)$	$Q_{2n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
59	66150	$(M-12)$	$Q_{2n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
60	−99225	$(M-12)$	$Q_{2n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	6
61	1470	$(M-12)$	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
62	−22050	$(M-12)$	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
63	66150	$(M-12)$	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	6
64	−11025	$(M-12)$	$Q_{2n}Q_{2n}Q_{2n}\|Q^*_{2n}Q^*_{2n}Q^*_{2n}$	6
65	−3920	$(M-12)$	$Q_{3n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
66	58800	$(M-12)$	$Q_{3n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6

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Table 7-continued from previous page

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
67	−176400	$(M-12)$	$Q_{3n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
68	58800	$(M-12)$	$Q_{3n}Q_nQ_nQ_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
69	−78400	$(M-12)$	$Q_{3n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
70	11760	$(M-12)$	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
71	−176400	$(M-12)$	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
72	529200	$(M-12)$	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
73	−176400	$(M-12)$	$Q_{3n}Q_{2n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
74	470400	$(M-12)$	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
75	−705600	$(M-12)$	$Q_{3n}Q_{2n}Q_n\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
76	−3920	$(M-12)$	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
77	58800	$(M-12)$	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
78	−176400	$(M-12)$	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
79	58800	$(M-12)$	$Q_{3n}Q_{3n}\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
80	−156800	$(M-12)$	$Q_{3n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
81	470400	$(M-12)$	$Q_{3n}Q_{3n}\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
82	−78400	$(M-12)$	$Q_{3n}Q_{3n}\|Q_{3n}^*Q_{3n}^*$	6
83	8820	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
84	−132300	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
85	396900	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
86	−132300	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
87	352800	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
88	−1058400	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
89	352800	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_{3n}^*Q_{3n}^*$	6
90	−396900	$(M-12)$	$Q_{4n}Q_nQ_n\|Q_n^*Q_n^*Q_{4n}^*$	6
91	−8820	$(M-12)$	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
92	132300	$(M-12)$	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
93	−396900	$(M-12)$	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
94	132300	$(M-12)$	$Q_{4n}Q_{2n}\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
95	−352800	$(M-12)$	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
96	1058400	$(M-12)$	$Q_{4n}Q_{2n}\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
97	−352800	$(M-12)$	$Q_{4n}Q_{2n}\|Q_{3n}^*Q_{3n}^*$	6
98	793800	$(M-12)$	$Q_{4n}Q_{2n}\|Q_n^*Q_n^*Q_{4n}^*$	6
99	−396900	$(M-12)$	$Q_{4n}Q_{2n}\|Q_{2n}^*Q_{4n}^*$	6
100	−14112	$(M-12)$	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	6
101	211680	$(M-12)$	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	6
102	−635040	$(M-12)$	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	6
103	211680	$(M-12)$	$Q_{5n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{2n}^*$	6
104	−564480	$(M-12)$	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{3n}^*$	6
105	1693440	$(M-12)$	$Q_{5n}Q_n\|Q_n^*Q_{2n}^*Q_{3n}^*$	6
106	−564480	$(M-12)$	$Q_{5n}Q_n\|Q_{3n}^*Q_{3n}^*$	6
107	1270080	$(M-12)$	$Q_{5n}Q_n\|Q_n^*Q_n^*Q_{4n}^*$	6

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Table 7-continued from previous page

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
108	−1270080	$(M-12)$	$Q_{5n}Q_n\|Q^*_{2n}Q^*_{4n}$	6
109	−1016064	$(M-12)$	$Q_{5n}Q_n\|Q^*_nQ^*_{5n}$	6
110	11760	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	6
111	−176400	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	6
112	529200	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	6
113	−176400	$(M-12)$	$Q_{6n}\|Q^*_{2n}Q^*_{2n}Q^*_{2n}$	6
114	470400	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_nQ^*_nQ^*_{3n}$	6
115	−1411200	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_{2n}Q^*_{3n}$	6
116	470400	$(M-12)$	$Q_{6n}\|Q^*_{3n}Q^*_{3n}$	6
117	−1058400	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_nQ^*_{4n}$	6
118	1058400	$(M-12)$	$Q_{6n}\|Q^*_{2n}Q^*_{4n}$	6
119	1693440	$(M-12)$	$Q_{6n}\|Q^*_nQ^*_{5n}$	6
120	−705600	$(M-12)$	$Q_{6n}\|Q^*_{6n}$	6
121	1	1	$Q_nQ_nQ_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
122	−42	1	$Q_{2n}Q_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
123	441	1	$Q_{2n}Q_nQ_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	7
124	210	1	$Q_{2n}Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
125	−4410	1	$Q_{2n}Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	7
126	11025	1	$Q_{2n}Q_{2n}Q_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	7
127	−210	1	$Q_{2n}Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
128	4410	1	$Q_{2n}Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	7
129	−22050	1	$Q_{2n}Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	7
130	11025	1	$Q_{2n}Q_{2n}Q_{2n}Q_n\|Q^*_nQ^*_{2n}Q^*_{2n}Q^*_{2n}$	7
131	140	1	$Q_{3n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
132	−2940	1	$Q_{3n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	7
133	14700	1	$Q_{3n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	7
134	−14700	1	$Q_{3n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_{2n}Q^*_{2n}Q^*_{2n}$	7
135	4900	1	$Q_{3n}Q_nQ_nQ_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{3n}$	7
136	−840	1	$Q_{3n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
137	17640	1	$Q_{3n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	7
138	−88200	1	$Q_{3n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	7
139	88200	1	$Q_{3n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_{2n}Q^*_{2n}Q^*_{2n}$	7
140	−58800	1	$Q_{3n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{3n}$	7
141	176400	1	$Q_{3n}Q_{2n}Q_nQ_n\|Q^*_nQ^*_nQ^*_{2n}Q^*_{3n}$	7
142	420	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_n$	7
143	−8820	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_nQ^*_{2n}$	7
144	44100	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_{2n}Q^*_{2n}$	7
145	−44100	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_nQ^*_{2n}Q^*_{2n}Q^*_{2n}$	7
146	29400	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_nQ^*_nQ^*_{3n}$	7
147	−176400	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_nQ^*_nQ^*_{2n}Q^*_{3n}$	7
148	44100	1	$Q_{3n}Q_{2n}Q_{2n}\|Q^*_{2n}Q^*_{2n}Q^*_{3n}$	7

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Table 7-continued from previous page

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
149	560	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
150	−11760	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
151	58800	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
152	−58800	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
153	39200	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
154	−235200	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
155	117600	1	$Q_{3n}Q_{3n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
156	78400	1	$Q_{3n}Q_{3n}Q_n\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
157	−420	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
158	8820	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
159	−44100	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
160	44100	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
161	−29400	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
162	176400	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
163	−88200	1	$Q_{4n}Q_nQ_nQ_n\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
164	−117600	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
165	44100	1	$Q_{4n}Q_nQ_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
166	1260	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
167	−26460	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
168	132300	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
169	−132300	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
170	88200	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
171	−529200	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
172	264600	1	$Q_{4n}Q_{2n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
173	352800	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
174	−264600	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
175	396900	1	$Q_{4n}Q_{2n}Q_n\|Q_n^*Q_{2n}^*Q_{4n}^*$	7
176	−840	1	$Q_{4n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
177	17640	1	$Q_{4n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
178	−88200	1	$Q_{4n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
179	88200	1	$Q_{4n}Q_{3n}\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
180	−58800	1	$Q_{4n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
181	352800	1	$Q_{4n}Q_{3n}\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
182	−176400	1	$Q_{4n}Q_{3n}\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
183	−235200	1	$Q_{4n}Q_{3n}\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
184	176400	1	$Q_{4n}Q_{3n}\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
185	−529200	1	$Q_{4n}Q_{3n}\|Q_n^*Q_{2n}^*Q_{4n}^*$	7
186	176400	1	$Q_{4n}Q_{3n}\|Q_{3n}^*Q_{4n}^*$	7
187	1008	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
188	−21168	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
189	105840	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7

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Table 7-continued from previous page

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
190	−105840	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
191	70560	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
192	−423360	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
193	211680	1	$Q_{5n}Q_nQ_n\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
194	282240	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
195	−211680	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
196	635040	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_{2n}^*Q_{4n}^*$	7
197	−423360	1	$Q_{5n}Q_nQ_n\|Q_{3n}^*Q_{4n}^*$	7
198	254016	1	$Q_{5n}Q_nQ_n\|Q_n^*Q_n^*Q_{5n}^*$	7
199	−1008	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
200	21168	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
201	−105840	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
202	105840	1	$Q_{5n}Q_{2n}\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
203	−70560	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
204	423360	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
205	−211680	1	$Q_{5n}Q_{2n}\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
206	−282240	1	$Q_{5n}Q_{2n}\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
207	211680	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
208	−635040	1	$Q_{5n}Q_{2n}\|Q_n^*Q_{2n}^*Q_{4n}^*$	7
209	423360	1	$Q_{5n}Q_{2n}\|Q_{3n}^*Q_{4n}^*$	7
210	−508032	1	$Q_{5n}Q_{2n}\|Q_n^*Q_n^*Q_{5n}^*$	7
211	254016	1	$Q_{5n}Q_{2n}\|Q_{2n}^*Q_{5n}^*$	7
212	−1680	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
213	35280	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
214	−176400	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
215	176400	1	$Q_{6n}Q_n\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
216	−117600	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7
217	705600	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
218	−352800	1	$Q_{6n}Q_n\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
219	−470400	1	$Q_{6n}Q_n\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
220	352800	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
221	−1058400	1	$Q_{6n}Q_n\|Q_n^*Q_{2n}^*Q_{4n}^*$	7
222	705600	1	$Q_{6n}Q_n\|Q_{3n}^*Q_{4n}^*$	7
223	−846720	1	$Q_{6n}Q_n\|Q_n^*Q_n^*Q_{5n}^*$	7
224	846720	1	$Q_{6n}Q_n\|Q_{2n}^*Q_{5n}^*$	7
225	705600	1	$Q_{6n}Q_n\|Q_n^*Q_{6n}^*$	7
226	1440	1	$Q_{7n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*$	7
227	−30240	1	$Q_{7n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_n^*Q_{2n}^*$	7
228	151200	1	$Q_{7n}\|Q_n^*Q_n^*Q_n^*Q_{2n}^*Q_{2n}^*$	7
229	−151200	1	$Q_{7n}\|Q_n^*Q_{2n}^*Q_{2n}^*Q_{2n}^*$	7
230	100800	1	$Q_{7n}\|Q_n^*Q_n^*Q_n^*Q_n^*Q_{3n}^*$	7

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Table 7-continued from previous page

i	x_i	$f_i^{(m=7,l)}$	Basis vectors	l
231	−604800	1	$Q_{7n}\|Q_n^*Q_n^*Q_{2n}^*Q_{3n}^*$	7
232	302400	1	$Q_{7n}\|Q_{2n}^*Q_{2n}^*Q_{3n}^*$	7
233	403200	1	$Q_{7n}\|Q_n^*Q_{3n}^*Q_{3n}^*$	7
234	−302400	1	$Q_{7n}\|Q_n^*Q_n^*Q_n^*Q_{4n}^*$	7
235	907200	1	$Q_{7n}\|Q_n^*Q_{2n}^*Q_{4n}^*$	7
236	−604800	1	$Q_{7n}\|Q_{3n}^*Q_{4n}^*$	7
237	725760	1	$Q_{7n}\|Q_n^*Q_n^*Q_{5n}^*$	7
238	−725760	1	$Q_{7n}\|Q_{2n}^*Q_{5n}^*$	7
239	−1209600	1	$Q_{7n}\|Q_n^*Q_{6n}^*$	7
240	518400	1	$Q_{7n}\|Q_{7n}^*$	7

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